GENERALIZED SPECIAL FUNCTIONS AND THEIR APPLICATIONS IN BOUNDARY VALUE PROBLEMS

THESIS
Submitted for the degree of
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS
TO
Bundelkhand University, Jhansi
1993

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CERTIFICATE

This is to certify that Mrs. Abha Tenguria actually carried out the work described in their thesis under my supervision at D.V. Post Graduate College, Orai. She has put the required attendance in the department during the period of her investigations.

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PREFACE

The present work is the outcome of the research carried out by me in the field of "Generalized Special Functions and Their Applications in Boundary Value Problems" at the Department of Mathematics, D.V. Post Graduate College, Orai, U.P., India.

This thesis consists of ten chapters, each divided into several sections (progressively numbered 1.1, 1.2, ...). The formulae are numbered progressively within each section. For example (8.2.4) denotes the 4th formula of second section in chapter VIII. References are given in alphabetic order at the end of each chapter.

The work was initiated in July 1989, under the able supervision of Dr. R.C. Singh Chandel, M. Sc., Ph. D., of D. V. Post Graduate College, Orai, U.P. I offer my heartfelt gratitude to my esteemed and generous guide Dr. Chandel for his guidance, keen interest, benevolent encouragement and valuable suuggestions at every step to carry out this work and critically going through manuscript.

I am sincerely thankful to Principal, Govt. M.L.B. Girls College, Bhopal and to the Principal of D.V. Post Graduate College, Orai, U.P. for the facilities that they have provided me during the tennure of research work.

I wish to express my thanks to all my friends, who helped me in multifacious ways throughout the present work.

I would not forget to use this opportunity in expressing my greatfulness to Mr. Anurag Tripathi of Vision Systems, Kanpur for meticulous, even tempered and consistently good judgement during the typing phase.

This work, however would be incomplete without the acknowledgement and encouragement of my affectionate husband and parents and other members of the family. I owe them much more than words can express for having borne patiently with my demands. I am also thankful to my cousin Mr. Alok Sharma for his efforts to bring out this manuscript. I also express my indebtedness to respected Mrs Chandel for her encouragement and inspiration.

Abha Tenguria

Place :
Date : 11.10.93

Mrs Abha Tenguria

List of Publications

1. *R.C. Singh Chandel and Abha Tiwari (1991)*
Multivariable analogue of Gould and Hopper's Polynomials defined by Rodrigues' formula. Indian J. Pure Appl. Math., 22 (9): 757-761.

2. *R.C. Singh Chandel and Abha Tiwari (1991)*
A Multivariable analogue of Hermite Polynomials. Ganita Sandesh 5 (2): 92-95.

3. *R.C. Singh Chandel and Abha Tiwari (1991)*
Generating relation involving hypergeometric functions of four variables. Pure Appl. Math. Sci. 34 (1-2): 15-25.

4. *R.C. Singh Chandel and Abha Tiwari (1990)*
Multiple hypergeometric function of Shrivastava and Daoust and its applications in two boundary value problems. Proc. VPI, 2, p.112.

5. *R.C. Singh Chandel and Abha Tiwari (1993)*
Multivariable analogue of Gould and Hopper's polynomials defined by Rodrigues' formula. (Presented in Third Annual Conference of VIJNANA PARISHAD OF INDIA held at H.N.B Garhwal University, Srinagar, U.P. on May 25- 26, 1993) (To appear in Bull. VPI, Vol.1, 1993)

6. *R.C. Singh Chandel and Abha Tiwari (1992)*
Another Multivariable analogue of Gould and Hopper's Polynomials. Pure Math. Manuscript Calcutta (In press)

7. *R.C. Singh Chandel and Abha Tiwari.*
A Multilinerar generating function. Mathematics Education (In press).

8. *R.C. Singh Chandel and Abha Tiwari 1993.*
Multiple hypergeometric function of Srivastava and Daoust and its applications in a problem involving Laplace equation Jñānābha, Vol.23, (To appear).

9. *R.C. Singh Chandel and Abha Tiwari.*
Multivariable analogue of Gould and Hopper's polynomials defined by Rodrigues' formula I. (Under communication with M.A.C.T. Journal Bhopal.)

10. *R.C. Singh Chandel and Abha Tiwari.*
Another multivariable analogue of Gould and Hopper's Polynomials defined by generating relation. (Under communication with 'Ganita Sandesh'.)

CONTENTS

CHAPTER-1

INTRODUCTION

In this chapter we give a brief historical account of some of the work done in the field of "Generalized Special Functions and Thier Applications in Boundary Value Problems". No attempt has been made to give a Comprehensive review of the entire literature on the subject but only those aspects, which have a direct bearing on our work done in the present thesis, have been dealt with in some details.

1.1 SPECIAL FUNCTIONS. An equation of the form

$$p_0(x)\,\omega^n + p_1(x)\,\omega^{n-1} + \ldots + p_n(x) = 0, \tag{1.1.1}$$

where $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials expressions having integral coefficients, is called algebraic equation.The roots of the above equation

$$w = f(x) \tag{1.1.2}$$

are called algebraic functions. The functions, which are not roots of algebraic equations are called "Transcendental Functions". Logarithmic functions, exponential functions, trigonometrical functions etc. are examples of "Transcendental Functions". Transcendental functions are generally solutions of differntial equations or they have integral representations. Transcendental functions such as beta functions, gamma functions, Bessel functions, E, G and H-functions, all polynomials etc., which are of complicated nature are known as "Higher Transcendental Functions".

In the study of Higher Transcendental Functions, if we are not concerned with their general properties, but only with the properties of the function which occur in the solution of special problems, they are called "Special Functions". Moreover, it is a matter of opinion or convension. According to Harry-Bateman (1882-1946) any function which has received individual attention at least in one research paper, may be attributed to Special Function.

Special Functions have several physical and Technical applications and also a continuously growing importance as they are connected with the general theory of orthogonal polynomials and related problems of mechanical quadrature.

Special Functions of Mathematical Physics arise in the solutions of partial differential equations governing the behaviour of certain physical quantities. The equations which occur frequently in pure and applied sciences are

(1.1.3) Wave equation $\Delta^2\,\phi = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$,

(1.1.4) Laplace equation $\Delta^2\,\phi = 0$,

and

(1.1.5) Diffusion equation $\Delta^2\,\phi = \dfrac{1}{k}\dfrac{\partial\phi}{\partial t}$,

where

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{1.1.6}$$

In each of the above equations t denotes time variable, c and k physical quantities which are generally constants and the function ϕ has to be determined. Its physical meaning depends upon the nature of

the problem. The equation (1.1.3) arises in problems which involve the phenomenon of wave motion and which occur in electromagnetism, acoustics, elasticity, hydrodynamics etc.

Equation (1.1.4) arises in potential problems, which occur in many branches of pure and applied sciences, viz. hydrodynamics, electrostatics, steady flow of heat and current, gravitation and elasticity. Equation (1.1.5) reduces to (1.1.4) when ϕ is time independent. It is general form it occurs in the theory of flow of heat, the skin effect for an alternating current in a conductor, in the theory of the transmission line and in certain diffusion problems. A wide range of physical problems are represented by the equations (1.1.3), (1.1.4) and (1.1.5).

There are various methods of solving these equations but one of the important methods, which is generally employed to solve them, is "Separation of Variables". The study of differential equations describing the physical situation and consistent with the boundary conditions, leads us to the Special Functions of Mathematical physics. Here we shall discuss some special functions, particularly, polynomials and their generalizations. We shall also discuss the hypergeometric functions in one, two , three, four and several variables.

1.2 Legendre function. Special Functions were first introduced towards the end of eighteenth century in the solution of the problems of Dynamical Astronomy and Mathematical physics. In 1782, Laplace introduced the potential theorem. Legendre (1782 or earlier) investigated the expansion of potential function in the form of an infinite series and was thus led to the discovery of functions now knows as "Legendre Coefficients" or Legendre polynomials.

Thomson and Tait in their well known "Natural Philosophy" (1879) defined spherical harmonics as follows :

Any function V_n of Laplace equation $\Delta^2 \phi = 0$, which is homogeneous of degree n in x, y, z is called a "Solid Spherical Harmonics of Degree n". The degree n may be any positive integer and the function need not be rational.

If x, y, z are expressed in terms of polar coordinates (r, θ, ϕ) the solid spherical harmonics of degree n assumes the form $r^n f_n(\theta, \phi)$. The function $f_n(\theta, \phi)$ is called a "Surface Spherical Harmonics of Degree n". Laplace equation possesses solutions of the form $\left\{ \begin{matrix} r^n \\ r^{-n-1} \end{matrix} \right\} e^{i m \phi} H(\mu)$, where $H(\mu)$ satisfies the ordinay differential equation

$$(1.2.1) \quad (1-\mu^2)\frac{d^2 H}{d\mu^2} - 2\mu \frac{dH}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} H = 0.$$

The above equation is called associated Legendre equation. μ is restricted to be a real and to lie in the interval (-1, 1).

Legendre polynomials were generalized by Gegenbauer Tchebicheff and Jacobi. Jacobi polynomials are most general polynomials of this family and were first introduced by C.G.Jacobi in 1859.

Jacobi polynomials (See Rainville [146, p.254, (1)]) are defined as

$$(1.2.2) \quad P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n ; \\ 1+\alpha ; \end{matrix} \; \frac{1-x}{2} \right],$$

For $\alpha = \beta = 0$, the above polynomials reduce to Legendre polynomials.

Generating function for Legendre polynomials is given by Rainville ([146, p.157(1)])

$$(1.2.3) \quad (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n,$$

while their Rodrigues' formula is given by Rainville ([146, p.162(7)])

(1.2.4) $$p_n(x) = \frac{1}{2^n\, n!} D^n (x^2 - 1).$$

1.3 HERMITE POLYNOMIALS. Hermite polynomials, first of all were discussed by Laplace in his two works: "Treatise on Celestial Mechanics" ([129], 1805) and "Theory of Probability" ([130], 1820). The systematic study of these polynomials was made by C.H. Hermite [112] in 1864. Hermite polynomials occur in case of the motion of the point mass in a field of force. Schrodinger [151] showed that a free particle which is represented by a wave function $\psi(\vec{r}, t)$, $\vec{r}$, being the position vector of the particle, satisfies the following differential equation :

(1.3.1) $$i\,h \frac{\partial \psi}{\partial t} = -\frac{h^2}{2m} \nabla^2 \psi ,$$

h being an universal constant. If the particle include the effect of the external forces such as electrostatic, gravitational, possibly nuclear which can be combined into a single force F, that is, derivable from the potential energy V, the above equation may be generalized into

(1.3.2) $$i\,h \frac{\partial \psi}{\partial t} = -\frac{h^2}{2m} \nabla^2 \psi + V(\vec{r}, t)\, \psi .$$

If the potential energy is independent of the time and $\psi(\vec{r}, t) = u(\vec{r})\, f(t)$, the equation may be seperated into

(1.3.3) $$\left[-\frac{h^2}{2m} \nabla^2 + V(\vec{r})\right] u(\vec{r}) = E\,.\,u(\vec{r}),$$

E being the seperation constant.

Further, the one dimensional motion of the point mass attracted to a fixed centre by a force proportional to the displacement from that centre, provides one of the fundamental problem of classical dynamics. The force F = -Kx can be represented by potential energy $V(x) = \frac{K x^2}{2}$, so that Schrodinger's equation in one dimension may be written in the form :

(1.3.4) $$-\frac{h^2}{2m} \frac{d^2 u}{d x^2} + \frac{1}{2} K x^2 = E\,.\,u\,.$$

Substituting $\alpha x = \xi$, the above equation becomes

$$\frac{d^2 u}{d \xi^2} + (\lambda - \xi^2)\, u. = 0\,.$$

We can find an exact solution of the above equation in the form :

(1.3.5) $$u(\xi) = H(\xi)\, e^{-\xi^2/2},$$

where $H(\xi)$ is a polynomial of finite order in ξ. This assumption on substitution into one dimensional equation leads to the differential equation

$$H''(\xi) - 2\xi H'(\xi) + (\lambda - 1) H(\xi) = 0.$$

In order to find the solution, choose $\lambda = 2n + 1$, so that

(1.3.6) $H_n''(\xi) - 2\xi H'_n(\xi) + 2n H_n(\xi) = 0$.

The function $H_n(\xi)$ is called Hermite polynomials of degree n in ξ.

Generating function for Hermite polynomials is given by Rainville ([146, p.187, (1)])

(1.3.7) $$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and their Rodrigues' formula is given by Rainville ([146, p.189, (2)])

(1.3.8) $H_n(x) = (-1)^n \exp(x^2) D^n(-x^2)$.

1.4 LAGUERRE POLYNOMIALS. E.de Laguerre [128] introduced Laguerre polynomials in 1879. These polynomials also occur in an unedited manuscript of Abel [1]. In physical problems these polynomials occur in case of the motion of two particles (nucleus and electron) that are attached to each other by a force that depends only on the distance between them.

The potential energy $V(r) = \frac{-ze^2}{r}$, which represents the attractive Coulomb interaction between an atomic nucleus of positive charge +ze and an electron of charge -e, provides a wave equation. The Schr/odinger wave equation describes the motion of a single particle in an external field. Now, however, we are interested in the motion of the two particles (nucleus and electron). The differential equation for the energy charecteristic state in this case is

(1.4.1) $$\frac{1}{2m_1}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial z_1^2}\right) + \frac{1}{2m_2}\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} + \frac{\partial^2 u}{\partial z_2^2}\right)$$
$$+ \frac{1}{h^2}\left[E_o - V\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right\}\right] u = 0 .$$

Subscripts 1 and 2 refer to first and second particles. Expressing the equation in terms of the coordinates of centre of mass and using the method of seperation of variables, the above equation gives

(1.4.2) $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[\frac{2\mu}{h^2}(E - V) - \frac{C}{r^2}\right] R = 0 ,$$

where C is constant of seperation. By suitably adjusting the constants and taking $R = \rho^l e^{-\rho/2} V(\rho)$, where $\rho = Kr$, the equation finally reduce to

(1.4.3) $$\frac{d^2 V}{d\rho^2} + \left[\frac{2(l+1)}{\rho} - 1\right] \frac{dv}{d\rho} + [c - (l+1)] \frac{V}{\rho} = 0 .$$

The physically acceptable solution of this equation with C = n may be represented in terms of Leguerre polynomials. These polynomials satisfy the following differential equation :

(1.4.4) $$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (1 + \alpha - x) \frac{d}{dx} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0 .$$

The generating function for Laguerre polynomials is given by Rainville [146, p.209(1)])

(1.4.5) $$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-1-\alpha} e^{-xt/1-t} ,$$

while their Rodrigues' formula is given by Rainville [146,p.205(5)])

(1.4.6) $$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} D^n \left[e^{-x} x^{\alpha+n}\right]$$

The polynomials stated above in (1.2), (1.3) and (1.4) are called classical orthogonal polynomials.

1.5 OTHER POLYNOMIALS. There are several hypergeometric polynomials which are non-orthogonal. In 1936 Bateman [9] was interested constructing inverse Laplace transforms. For this purpose he introduced the polynomials

(1.5.1) $$Z_n(x) = {}_2F_2(-n, n+1; 1, 1; x) .$$

Rice [147] made a considerable study of the polynomials defined by

(1.5.2) $$H_n(\zeta, p, v) = {}_3F_2\left[\begin{matrix} -n, n+1, \zeta; \\ p, 1; \end{matrix} v\right]$$

Bateman [8] studied the polynomials

(1.5.3) $$F_n(z) = {}_3F_2\left[-n, n+1, \frac{1}{2}(1+z); 1, 1; z\right],$$

quite extensively, and which were generalized by Pasternak in the following way :

(1.5.4) $$F_n^{(m)}(z) = F\left[\begin{matrix} -n, n+1, \frac{1}{2}(1+z+n); \\ 1, \quad m+1; \end{matrix} 1\right].$$

Another polynomial, in which the interest is concentrated on a parameter, is Mittage-Leffer polynomial.

(1.5.5) $$g_n(z) = 2Z\,{}_2F_1[1-n, 1-z; 2; 2].$$

Bateman (1940) generalized the above polynomials in the form :

(1.5.6) $$g_n(z, r) = \frac{(-r)_n}{n!}\,{}_2F_1(-n, z; -r; 2) .$$

Sister Celine (Fesenmyer [104]) concentrated on the polynomials generated by

(1.5.7) $$(1-t)^{-1}\,{}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \frac{-4xt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1, \ldots, a_p; \\ 1, \quad \frac{1}{2}, \quad b_1, \ldots, b_q; \end{matrix} x\right] t^n$$

Her polynomials include Legendre polynomials, some special Jacobi, Rice's $H_n(\zeta, p, v)$, Bateman's $Z_n(x)$, $F_n(z)$ and Pasternak's polynomials etc. as special cases.

1.6 ORTHOGONAL POLYNOMIALS. If $\{\varphi(x)\}$ denotes a sequence of functions with a weight function w(x) in an interval (a, b) which is non-negative there, we may associate the scalar product

(1.6.1) $$(\varphi_1, \varphi_2) = \int_a^b \omega(x)\,\varphi_1(x)\,\varphi_2(x)\,dx ,$$

which is defined for all function φ for which $\omega^{1/2}\varphi$ is quadratically integrable in (a, b). Two functions are said to be orthogonal if their scalar product vanishes.

The function $\{\phi_n(x)\}$, thus form an orthogonal system if

(1.6.2) $$(\phi_n, \phi_k) \begin{matrix} = 0 & \text{if } h \neq k \\ \neq 0 & \text{if } h = k \end{matrix} .$$

It is well known that Legendre, Gegenbauer, Jacobi, Hermite and laguerre polynomials each form an orthogonal set. These polynomials arrise very frequently. They have number of common properties. The following four of which are most important :

(i) $\{\phi'_n(x)\}$ is a system of orthogonal polynomials.

(ii) $\phi_n(x)$ satisfy a differential equation of the form

(1.6.3) $$A(x)\, y'' + B(x)\, y' + \lambda_n\, y = 0 ,$$

where A(x) and B(x) are independent of n and λ_n is independent of x ,

(iii) There is generalized Rodrigues' formula

(1.6.4) $$\phi_n(x) = \frac{1}{k_n\, \omega(x)} \frac{d^n}{d x^n}\left[\omega(x)\, x^n\right] ,$$

where k_n is constant and x is a polynomial in x whose coefficients are independent of n.

(iv) Any real polynomials set $\{ f_n(x) \}$ which satisfies a three term recurrence relation

(1.6.5) $$x\, f_n(x) = A_n\, f_{n+1}(x) + B_n\, f_n(x) + C_n\, f_{n-1}(x)$$

where $A_n \neq 0$, $C_n \neq 0$, is orthogonal with respect to some weight function, over some interval. This well known property is due to Favard[103].

Other important proprties of orthogonal polynomials are the self adjoint form of the differential equation

(1.6.6) $$\frac{d}{d x}\left[x\, \omega(x) \frac{d y}{d x}\right] + \lambda_n\, \omega(x)\, y = 0,$$

and the Chirstoffel-Darboux formula

(1.6.7) $$\sum_{r=0}^{n} A_r\, \phi_r(x)\, \phi_r(y) = B_n\left[\frac{\phi_{n+1}(x)\, \phi_n(y) - \phi_n(x)\, \phi_{n+1}(y)}{x - y}\right] .$$

For orthogonal polynomials see also Chandel [28].

1.7 HYPERGEOMETRIC FUNCTION OF ONE VARIABLE.

The Gaussion Hypergeometric series. In the study of second order linear differential equations with three regular singular points there arises the function

(1.7.1) $${}_2F_1(a, b; c; z) = {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n} \frac{z^n}{n!} , \quad c \neq 0, -1, -2, \ldots$$

The above infinite series obviously reduces to the elementary geometric series

(1.7.2) $$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + z^n + \ldots$$

in the special case when

(1.7.3) (i) $a = c$ and $b = 1$ (ii) $a = 1$ and $b = c$.

Hence it is called hypergeometric series or more precisely, gauss's hypergeometric series after the famous German mathematician Carl Friedrich Gauss (1777-1855), who in the year 1812 introduced this series into analysis and gave the F- notation for it.

By d' Alembert's ratio test, it is easily seen that the hypergeometric series in (1.7.1) converges absolutely within the unit circle, that is, when $|z| < 1$, provided that the denominator parameter c is neither zero nor negative integer. However, we notice if either or both of the numerator parameters a and b in (1.7.1) is zero or negative integer, the hypergeometric series terminates and the series is automatically convergent. Further tests readily show that the hypergeometric series in (1.7.1) when $|z| = 1$ (that is, on the unit circle), is

(i) absolutely convergent if $\mathrm{Re}\,(c-a-b) > 0$,

(ii) Conditionally convergent if -1 $\mathrm{Re}\,(c-a-b) \leq 0$, $z \neq 1$,

(iii) divergent if $\mathrm{Re}\,(c-a-b) \leq -1$

In case (i), for a number of summation theorems for the hypergeometric series (1.7.1) when z takes on other special values, see Bailey ([7], 1935, pp.9-11), Erdelyi et al. ([94], 1953, pp.104-105), Slater ([162], 1966, p.243), Luke ([131], 1975, pp.271-273) and Srivastava-Manocha ([179], 1984, pp.29-31).

GENERALIZED HYPERGEOMETRIC SERIES. A natural generalization of above Gaussion Hypergeometric series ${}_2F_1$ (a, b; c; z) is accomplished by introducing any arbitrary number of numerator and denominator parameters. The resulting series

(1.7.7)
$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p; \\ b_1,\dots,b_q; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} $$
$$ = {}_pF_q\,(a_1,\dots,a_p\,;\; b_1,\dots,b_q\,;\; z) $$

is known as the generalized Gauss series, or simply, the generalized hypergeometric series. Here p and q are positive integers or zero (interpreting an empty product as 1), we assume that the variable z, the numerator parameters $a_1,\dots,a_p$, and denominator parameters $b_1,\dots,b_q$ take on complex values, provided that

(1.7.8) $b_j \neq 0, -1, -2, \dots;\; j = 1,\dots, q.$

Supposing that none of the numerator parameters is zero or negative integer (otherwise question of convergence will not arise , and with usual restriction (1.7.8) the ${}_pF_q$ series in (1.7.7)

(i) converges for $|z| < \infty$ if $p \leq q$

(ii) converges for $|z| < 1$ if $p = q + 1$ and

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$

Further more, if we set

(1.7.9)
$$ \omega = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j , $$

then the series ${}_pF_q$ with $p = q + 1$, is

(i) absolutely convergent for $|z| = 1$ if $\mathrm{Re}\,(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1, z \neq 1$ if -1 $\mathrm{Re}(\omega) \leq 0$, and

(iii) divergent for $|z| = 1$, if $Re(\omega) \le -1$.

1.8 A FURTHER GENERALIZATIOIN OF $_pF_q$. An interesting further generalization of the series $_pF_q$ is due to Fox [105] and Wright ([185], [186]), who studied asymptotic expansion of the generalized hypergeometric function defined by

$$(1.8.1) \qquad {}_p\Psi_q\left[\begin{matrix}(a_1, A_1),\ldots,(a_p, A_p)\,;\\(b_1, B_1),\ldots,(b_q, B_q)\,;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j + A_j n)}{\prod_{j=1}^{q} (b_j + B_j n)} \frac{z^n}{n!}$$

where the coefficients $A_1,\ldots, A_p$ and $B_1,\ldots, B_q$ are positive real numbers such that

$$(1.8.2) \qquad 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \ge 0\,.$$

By comparing (1.7.7) and (1.8.1), we have

$$(1.8.3) \qquad {}_p\Psi_q\left[\begin{matrix}(a_1, 1),\ldots,(a_p, 1)\,;\\(b_1, 1),\ldots,(b_q, 1)\,;\end{matrix}\; z\right] = \frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(b_j)}\; {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\b_1,\ldots,b_q;\end{matrix}\; z\right].$$

1.9 **HYPERGEOMETRIC SERIES IN TWO VARIABLES.**

The great success of the hypergeometric series in one variable has stimulated the development of a corresponding theory in two or more variables. Appell [4] has defined four double hypergeometric series F_1, F_2, F_3, F_4, (known as Appell series), analogus to Gauss's $_2F_1$ (a, b; c; z) . The standard work on the theory of Appell series is the monograph by Appell and Kampé de Fériet [6], which contains an extensive bibliography of all relevant papers upto 1926 (by for example, L.Pochhammer, J.Horn, E.Picard, E.Goursat). See Erdélyi et al. [94, pp. 222-245] for a review of a subsequent work on the subject; see also Bailey ([7], chapter 9), Slater ([162], chapter 8) and Exton ([101], pp. 23-28). Horn puts

$$f(m,n) = \frac{F(m, n)}{F'(m, n)}\,, \quad g(m, n) = \frac{G(m, n)}{G'(m, n)}\,,$$

where F, F', G, G' are polynomials in m, n of respective degrees p, p', q, q, F' is assumed to have factor m + 1 ,and G' a factor n + 1 ; F and F' have no common factor except possibily, m + 1; and G and G' have no common factor except possibily n + 1 . The greatest of the four numbers p,p' , q,q' is the order of the hypergeometric series. Horn investigated, in particular, the hypergeometric series order two and found that , apart from certain seies which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially thirty four distinct convergent series of order two (Horn [114], correction in Borngässer [15]).

Horn Series.

Horn [114] defined the ten hypergeometric series in two variables and denoted them by G_1, G_2, G_3, $H_1,\ldots, H_7$; he thus completed the set of all fourteen possible second order (complete) hypergeometric series in two variables Appell and Kampé de Fériet ([6], p. 143 et seq.), see also Erdelyi et al. ([94]. pp. 224-228).

Cofluent Hypergeometric series in Two variables.

Seven confluent forms of the four Appell series were defined by Humbert [115] and he denoted these confluent hypergeometric series in two variables by ϕ_1, ϕ_2, ϕ_3, Φ_1, Φ_2, Φ_3, E_1, E_2,.

In addition, there exist thirteen confluent forms of the Horn series which are denoted by Horn [114] and Borngässer [15] Γ_1, Γ_2, $H_1, ..., H_{11}$. Thus there are twenty possible confluent hypergeometric series in two variables.

The work of Humbert has been described reasonable fully by Appell and Kampé de Fériet ([6]. pp.124-135), and the definitions and convergence conditions of all these twenty confluent hypergeometric series in two variables are given also in Erdélyi et al ([94], pp. 225-228).

For more details see Srivastava and Karlsson [180].

Kampé de Fériet Series and Its Generalization.

Just as the Gaussion series ${}_2F_1$ was generalized to ${}_pF_q$ by increasing the numbers of numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [120], who defined a general hypergeometric series in two variables (see Appell and Kampé de Fériet [6, p.150 (29)]). The notation introduced by Kampé de Fériet [loc. cit] for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy ([16], p.112).

A further generalization of the Kampé de Fériet series is due to Srivastava and Daoust ([170], 1969), who indeed defined the extension of the ${}_p\psi_q$ series (1.8.3) in two variables.

Later on in 1976, a generalization of Kampé de Fériet series is also seen in the literature due to Srivastava and Panda [(176), p.423,(26)] but it is special case of Srivastava and Daoust ([170], 1969).

1.10 Triple Hypergeometric Seires. Laurícella [127, p. 114] introduced fourteen complete hypergeometric series in three variables of the second order. He denoted his triple hypergeometric series by the symbols $F_1, F_2, F_3, ..., F_{14}$ of which four series F_1, F_2, F_5 and F_9 correspond respectively respectively to the three variable Lauricella series $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$, and $F_D^{(3)}$.

The remaining ten series F_3, F_4, F_6, F_7, F_8, F_{10}, ..., F_{14} of Lauricella's set apparently fell into oblivion except that there is an isolated appearance of the triple hypergeometric series F_8 in a paper by Mayr [135,p.265] who came across this series while evaluating certain infinite integrals. Saran [150] initiated a systematic study of these ten triple hypergeometric series of Lauricella's set. Saran's notations are F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S and F_T for the series F_4, F_{14}, F_8, F_3, F_{11}, F_6, F_{12}, F_{10}, F_7, F_{13} respectively (see also Chandel [27]).

Srivastava Triple Hypergeometric series H_A, H_B, H_C.

In the course of further investigation of Lauricella's fourteen hypergeometric series in three variables, Srivastava ([163], [164], [168]) noticed the existence of three additional complete triple hypergeometric series of the second order. These three series H_A, H_B and H_C had been neither included in the Lauricella's set, nor were they previously mentioned in the literature. H_C is new and interesting generalization of Appell's series F_1; H_B generalizes the Appell series F_2, while H_A provides a generalization of both F_1 and F_2.

A unification of Lauricella's fourteen hypergeometric series $F_1, ..., F_{14}$ and the additional series H_A, H_B, H_C was introduced by Srivastava [167,p.428], who defined general triple hypergeometric series.

While transforming Pochhammer's double-loop counter integrals associated with the series F_8 and F_{14} (i.e. F_G and F_F, respectively) belonging to Lauricella's set of hypergeometric series in three variables, the two interesting triple hypergeometric series G_A, G_B of Horn's type were encountered by Pandey ([140], pp.115-116). An investigation of the system of partial differential equation associated with the triple hypergeometric series H_C of Srivastava ([163], [164], [168]) led Srivastava [169,p.105 (3.5)] to the new series G_C. Other triple hypergeometric series studied in the literature are introduced by Dhawan [92], Samar [149] and Exton ([100], [102]).

1.11 The Quadruple HypergeometricFunctions. Until the Exton [98] defined and examined a few of their properties, no specific study had been made of any hypergeometric function of four variables apart from the four Lauricella's function $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$ and $F_D^{(4)}$ and certain of their

limiting cases. On account of the large number of such functions which arises from a systematic study of all the possibilities he restricted himself to those functions which are complete and of the second order and which involve at least one product of the type (a, k + m + n + p), in series representation; k, m, n, p are indices of quadruple summation. Exton ([98], [101]) defined following twenty one quadruple hypergeometric series, which will be used in our investigations (Chapter VI) :

(1.11.1) $K_1(a, a, a, a; b, b, b, c; d, e_1, e_2, d; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m+n)(c, p)}{(a, k+p)(e_1, m)(e_2, n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.2) $K_2(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, t)$

$$= \sum \frac{a, k+m+n+p)(b, k+m+n)}{(d_1, k)(d_2, n)(d_3, n)(d_4, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.3) $K_3(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k+m)(b_2, n+p)}{(c_1, k+p)(c_2, m+n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.4) $K_4(a, a, a, a; b_1, b_1, b_2, b_2; c, d_1, d_2, c; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k+m)(b_2, n+p)}{(c, k+p)(d_1, m)(d_2, n)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.5) $K_5(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_3, c_4; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k+m)(b_2, n+p)}{(c_1, k)(c_2, m)(c_3, n)(c_4, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.6) $K_6(a, a, a, a; b, b, c_1, c_2; c, d, d, d; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m)(c_1, n)(c_2, p)}{(c, k)(d, m+n+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.7) $K_7(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m)(c_1, n)(c_2, p)}{(d_1, k+n)(d_2, m+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.8) $K_8(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m)(c_1, n)(c_2, p)}{(d, k+m)(e_1, m)(e_2, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.9) $K_9(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m)(c_1, n)(c_2, p)}{(e_1, k)(e_2, m)(d, n+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.10) $K_{10}(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_3, d_4; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b, k+m)(c_1, n)(c_2, p)}{(d_1, k)(d_2, m)(d_3, n)(d_4, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.11) $K_{11}(a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, d; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k)(b_2, m)(b_3, n)(b_4, p)}{(c, k+m+n)(d, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.12) $K_{12}(a, a, a, a; b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k)(b_2, m)(b_3, n)(b_4, p)}{(c_1, k+m)(c_2, n+p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.13) $K_{13}(a, a, a, a; b_1, b_2, b_3, b_4; c, c, d_1, d_2; x, y, z, t)$

$$= \sum \frac{(a, k+m+n+p)(b_1, k)(b_2, m)(b_3, n)(b_4, p)}{(c, k+m)(d_1, n)(d_2, p)} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^p}{p!},$$

(1.11.14) $K_{14}(a, a, a, c_3; b, c_1, c_2, b; d, d, d, d; x, y, z, t)$

$$= \sum \frac{(a, k+m+n)(c_3, p)(b, k+p)(c_1, m)(c_2, n)}{(d, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.15) $K_{15}(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, t)$

$$= \sum \frac{(a, k+m+n)(b_5, p)(b_1, k)(b_2, m)(b_3, n)(b_4, p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.16) $K_{16}(a_1, a_2, a_3, a_4; b; x, y, z, t)$

$$= \sum \frac{(a_1, k+m)(a_2, k+n)(a_3, m+p)(a_4, n+p)}{(b, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.17) $K_{17}(a_1, a_2, a_3, b_1, b_2; c; x, y, z, t)$

$$= \sum \frac{(a_1, k+m)(a_2, k+n)(a_3, m+n)(b_1, p)(b_2, p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.18) $K_{18}(a_1, a_2, a_3, b_1, b_2; c; x, y, z, t)$

$$= \sum \frac{(a_1, k+m)(a_2, k+p)(a_3, m+n)(b_1, n)(b_2, p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.19) $K_{19}(a_1, a_2, b_1, b_2, b_3, b_4; c; x, y, z, t)$

$$= \sum \frac{(a_1, k+m)(a_2, k+n)(b_1, m)(b_2, n)(b_3, p)(b_4, p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(1.11.20) $K_{20}(a_1, a_1, b_3, b_4; b_1, b_1, a_2, a_2; c, c, c, c; x, y, z, t)$

$$= \sum \frac{(a_1, k+m)(b_3, n)(b_4, p)(b_1, k)(b_2, m)(a_2, n+p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

and

(1.11.21) $K_{21}(a, a, b_6, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, t)$

$$= \sum \frac{(a, k+m)(b_6, n)(b_5, p)(b_1, k)(b_2, m)(b_3, n)(b_4, p)}{(c, k+m+n+p)} \frac{x^k}{k!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^p}{p!},$$

(see also Chandel and Dwivedi [62]).

Recently Sharma and Parihar [154] introduced eighty three hypergeometric functions of four variables. It is worthly to note that out of these eighty three functions, nineteen functions had already been included in the set of 21 functions introduced by Exton ([98], [101]) in different notation (see, Remark due to Chandel and Kumar [74]). Further very recently Chandel, Agarwal and Kumar [46] have also introduced seven more hypergeometric functions of four variables.

1.12 MULTIPLE HYPERGEOMETRIC SERIES OF SEVERAL VARIABLES.

While several authors, for example, Green [107], Hermite [113] and Dedon [93] have discussed what amount to certain specified hypergeometric functions. It was left to Lauricella [127] to approach this topic systematically. Begining with the Appell functions Lauricella proceeded to define and study the four important functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ which bear his name. A number of confluent forms of the above Lauricella's functions denoted by $\phi_2^{(n)}$ and $\psi_2^{(n)}$ exist in the literature (for instance see Erdélyi [96,p.446(7.2)]; Humbert [117,p.429], see also Appell and Kampé de Fériet [6,p.134(34)], Chandel [29] and Chandel- Dwivedi [60]). Some other confluent forms of Lauricella series have appeared in the literature. These include the confluent series $\phi_D^{(n)}$ introduced by Srivastava and Exton [172,p.373(12)] and confluent series $E_1^{(n)}$ and $\phi_3^{(n)}$ used by Exton [101,p.43, (2.1.1.4), (2.1.1.5)].

Generalization of Lauricella's Series. An interesting unification and generalization) of Lauricella's multiple series $F_A^{(n)}$ and $F_B^{(n)}$ and Horn's double series H_2 was considered by Erdélyi [95,p.13, (28)], He denoted his series by $H_{n,p}$.

Srivastava and Daoust [171,p.454] (also see Srivastava and Manocha [179,p.64, (18),(19),(20)]) considered a multivariable extension of the series ${}_p\psi_q$ defined by (1.8.3). Their multiple hypergeometric series, known as the generalized Lauricella series in several variables is defined as

$$(1.12.1)\qquad S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix}$$

$$= S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: & [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}]: & [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\dots,x_n\right]$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma\left(a_j+\sum_{i=1}^{n}m_i\theta_j^{(i)}\right)\prod_{j=1}^{B'}\Gamma(b'_j+m_1\phi'_j),\dots,\prod_{j=1}^{B^{(n)}}\Gamma\left(b_j^{(n)}+m_n\phi_j^{(n)}\right)}{\prod_{j=1}^{C}\Gamma\left(c_j+\sum_{i=1}^{n}m_i\psi_j^{(i)}\right)\prod_{j=1}^{D'}\Gamma(d'_j+m_1\delta'_j),\dots,\prod_{j=1}^{D^{(n)}}\Gamma\left(d_j^{(n)}+m_n\delta_j^{(n)}\right)}\ \frac{x_1^{m_1}}{m_1!},\dots,\frac{x_n^{m_n}}{m_n!}$$

or alternatively by

$$(1.12.2)\quad F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]: & [(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}]: & [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\dots,x_n\right]$$

$$=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j,m_1\theta'_j+\dots+m_n\theta_j^{(n)})\prod_{j=1}^{B'}(b'_j,m_1\theta'_j)\dots\prod_{j=1}^{B^{(n)}}(b_j^{(n)},m_n\phi_j^{(n)})}{\prod_{j=1}^{C}(c_j,m_1\psi'_j+\dots+m_n\psi_j^{(n)})\prod_{j=1}^{D'}(d'_j,m_1,\delta'_j)\dots\prod_{j=1}^{D^{(n)}}(d_j^{(n)}),m_n\delta_j^{(n)})}\ \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

where

$$(1.12.3)\qquad \theta_j^{(i)},\ j=1,\dots,A;\ \phi_j^{(i)},\ j=1,\dots,B^{(i)};\ \psi_j^{(i)},\ j=1,\dots,C;$$

$$\delta_j^{(i)},\ j=1,\dots,D^{(i)};\quad 1\le i\le n;$$

are real and positive and (a) is taken to abbreviate the sequence of A parameters $a_1,\dots,a_A$; $b^{(i)}$ abbreviates the sequence of $B^{(i)}$ parameters $b_1^{(i)},\dots,b_{B^{(i)}}^{(i)}$, $i=1,\dots,n$; with similar interpretations for (c) and $(d^{(i)})$, $i=1,\dots,n$; etc.. For $n=2$, the above series reduces to the series defined by Srivastava and Daoust [170]. For more details see Chandel and Dwivedi [61] and Chandel and Gupta [67]. The above series will be frequently used in our investigations.

Some Other Generalization of Lauricella's Series. Some other interesting generalization of Lauricella Series studied in the literature, include the two multiple hypergeometric series ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(2)}E_D^{(n)}$ related to Lauricella's $F_D^{(n)}$ introduced by Exton ([99], [101]). Prompted by this work

Chandel [31] defined and studied a multiple hypergeometric function ${}^{(k)}_{(1)}E^{(n)}_{C}$ closely related to Lauricella's $F^{(n)}_{C}$.

Generalization of Horn Series. Exton [100, p.163(4.5)] introduced a multiple hypergeometric series $D^{p,q}_{(n)}$, which for p = q reduces to Exton ([97], p.86 for p = 1, 2,...; see also [101], p.104, (3.6.1)), which provides a multivariable generalization of the Horn series G_2. Exton [101] considered three other generalizations by ${}^{(p)}H^{(n)}_{j}$, j = 2, 3, 4. of these multivariable Horn series ${}^{(p)}H^{(n)}_{2}$ is simply Erdélyi's series $H_{n,p}$ [95, p.13,(28)] and for remaining two generalizations one may refer to Exton ([101], p.97,(3.5.1) and (3.5.2)).

INTERMEDIATE LAURICELLA'S FUNCTION. By taking a commendable idea of interpolation between Lauricella's function, Chandel and Gupta [70] introduced three multiple hypergeometric functions ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$ and ${}^{(k)}F^{(n)}_{BD}$ related to Lauricella's function. We have the following interesting relationships :

$$(1.12.4)\quad {}^{(0)}F^{(n)}_{AC} = F^{(n)}_{A},\quad {}^{(1)}F^{(n)}_{AC} = F^{(n)}_{A},\quad {}^{(n)}F^{(n)}_{AC} = F^{(n)}_{C}$$

$$(1.12.5)\quad {}^{(0)}F^{(n)}_{AD} = F^{(n)}_{A},\quad {}^{(1)}F^{(n)}_{AD} = F^{(n)}_{A},\quad {}^{(n)}F^{(n)}_{AD} = F^{(n)}_{D}$$

and

$$(1.12.6)\quad {}^{(0)}F^{(n)}_{BD} = F^{(n)}_{B},\quad {}^{(1)}F^{(n)}_{BD} = F^{(n)}_{B},\quad {}^{(n)}F^{(n)}_{BD} = F^{(n)}_{D}\,.$$

Chandel and Gupta [70] also introduced five confluent forms

${}^{(k)}_{(1)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{AC}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{AD}$, ${}^{(k)}_{(1)}\Phi^{(n)}_{BD}$ and ${}^{(k)}_{(2)}\Phi^{(n)}_{BD}$ of their above series.

Prompted by this work Karlsson [121] also introduced the fourth possible intermediate Lauricella function ${}^{(k)}F^{(n)}_{CD}$ for which we have

$$(1.12.7)\quad {}^{(0)}F^{(n)}_{CD} = F^{(n)}_{C},\quad {}^{(n)}F^{(n)}_{CD} = F^{(n)}_{D}.$$

Recently, Chandel and Vishwakarma ([82], [83]) introduced and studied many confluent forms of the above series.

No doubt, the series

${}^{(k)}_{(1)}E^{(n)}_{D}$, ${}^{(k)}_{(2)}E^{(n)}_{D}$, ${}^{(k)}_{(1)}E^{(n)}_{C}$, ${}^{(p)}H^{(n)}_{3}$, ${}^{(p)}H^{(n)}_{4}$, ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$, ${}^{(k)}F^{(n)}_{CD}$
and their confluent forms are include in the generalized Lauricella series of Srivastava and Daoust defined by (1.12.1) or (1.12.2), but they have their own importance.

1.13 EXTENSION OF MOST GENERALIZED HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST.

As natural further generalization of the (Srivastava - Daoust) generalized Lauricella function of several complex variables defined by (1.12.1) or (1.12.2), H-function of two variables of Mittal-Gupta [138] and G-function of two variables of Agarwal [3] (also see Chandel-Agrawal [42]), is given by Srivastava and Panda ([174], p.271, (4.1); [175], p.121, (1.10)) by means of the multiple contour integral

$$(1.13.1)\quad H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\ldots;[(b^{(r)}):\varphi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\varphi_1(\zeta_1)\ldots\varphi_r(\zeta_r)\,\psi(\zeta_1,\ldots,\zeta_r)\,z_1^{\zeta_1}\ldots z_r^{\zeta_r}\,d\zeta_1\ldots d\zeta_r,\quad \omega=\sqrt{-1}$$

where

(1.13.2) $$\varphi_i(\zeta_i) = \frac{\prod_{j=1}^{\mu^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}\zeta_i)\prod_{j=1}^{\nu^{(i)}}\Gamma(1-b_j^{(i)}+\varphi_j^{(i)}\zeta_i)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\zeta_i)\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\varphi_j^{(i)}\zeta_i)},\quad \forall\, i\in\{1,\cdots,r\};$$

(1.13.3) $$\psi(\zeta_1,\ldots,\zeta_r) = \frac{\prod_{j=1}^{\lambda}\Gamma(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i)}{\prod_{j=\lambda+1}^{A}\Gamma(a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i)\prod_{j=1}^{C}\Gamma(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i)}$$

an empty product is interpreted as 1, the coefficients, $\theta_j^{(i)}, j = 1,\ldots, A$; $\phi_j^{(i)}, j = 1,\ldots, B^{(i)}$; $\psi_j^{(i)}, j = 1,\ldots, C$; $\delta_j^{(i)}, j = 1,\ldots, D^{(i)}$ $\forall i \in \{1,\ldots, r\}$ are positive numbers, and $\lambda, \mu^{(i)}, \nu^{(i)}, A, B^{(i)}, C, D^{(i)}$ are integers such that $0 \le \lambda \le A$, $0 \le \mu^{(i)} \le D^{(i)}$, $C \ge 0$, and $0 \le \nu^{(i)} \le B^{(i)}$, $\forall i \in \{1,\ldots, r\}$. The contour L_i in the complex ζ_i - plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)})$, $j = 1,\ldots, \mu^{(i)}$, are to the right, and those of $\Gamma(1 - b_j^{(i)} + \phi_j^{(i)}\zeta_i)$, $j = 1,\ldots, \nu^{(i)}$ and $(1 - a_j + \sum_{i=1}^{r}\theta_j^{(i)}\zeta_i)$, $j = 1,\ldots, \lambda$, to the left, of L_i, the various parameters being so restricted that these poles are all simple and none of them coincide; and with the points $z_i = 0, \forall i \in \{1,\ldots, r\}$, being tacitly excluded, the multiple integrals in (1.13.1) converges absolutely if

(1.13.4) $$|\arg z_i| < \frac{1}{2}\pi\Delta_i, \forall i \in \{1,\ldots, r\},$$

where

(1.13.5) $$\Delta_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0, \forall i \in \{1,\ldots, r\}.$$

The above function is most generalized function of several complex variables and it will be used in the last chapter of our thesis.

1.14 GENERALIZATION AND UNIFIED PRESENTATION OF POLYNOMIALS.

The orthogonal and non-orthogonal polynomials may be generalized in four ways ; (i) by suitable generating function (ii) by Rodrigues' formula (iii) by recurrence relation and (iv) by differential equation. In this present thesis we shall make appeal to first two methods.

(i) By Defining Suitable Generating Function. The name "Generating Function" was first introduced by Laplace [130] in 1812. If a function F(x, t) has a power series (not necessarily convergent) expansion in t, and it is of the form

$$(1.14.1) \qquad F(x, t) = \sum_{n=0}^{\infty} a_n f_n(x) t^n ,$$

where a_n; n = 0, 1, 2,... be specified sequence independent of x and t then F(x, t) is called generating function of $f_n(x)$.

In the study of polynomial sets, there is a great importance of generating functions. For the use of generating functions we may refer to Sheffer [152], Brenke [14], Rainville ([145], [146]), Huff [118], Truesdell [183], Palas [139], Boas and Buck [13], Zeitlin [187] and Gould-Hopper [109] etc.. Recently Mittal ([136], [137]) and Panda [144] have also discovered many interesting and useful generating functions and operational generating functions for a large number of special functions (polynomials) of Laguerre, Hermite, Bessel, Jacobi etc.

Singhal and Srivastava [155] studied a class of bilateral generating functions for certain classical polynomials. Also Srivastava-Lavoie [173] and Srivastava [166] presented a systematic introduction to and several applications of general method of obtaining bilinear, bilateral or mixed multilateral generating functions for a fairly wide veriety of special functions in one, two or more variables, Bhargava [11] used their theorems for obtaining some bilinear, bilateral and mixed multilateral generating functions. For more details of Generating functions see Chandel-Yadava ([88], [90], [91]), Chandel-Sahgal [76] and Srivastava and Manocha [179].

In 1947, Fasenmyer [104] studied the polynomials (called Sister Celine's polynomials) generated by

$$(1.14.2) \qquad (1-t)^{-1} \, {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p ; \\ b_1, \ldots, b_q ; \end{matrix} \; \frac{-4xt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} {}_{p+2}F_{q+2} \left[\begin{matrix} -n, \; n+1, a_1, \ldots, a_p ; \\ 1, \; 1/2, b_1, \ldots, b_q ; \end{matrix} \; x\right] t^n .$$

Her polynomials include as special cases the Legendre polynomials $P_n(1-2x)$, Jacobi polynomials, Rice's H_n (p, q, x), Bateman's polynomials $Z_n(x)$ and $F_n(x)$. For generalized Rice polynomials see Chandel and Pal [75]. Chandel ([23] to [26]) studied the generalized Laguerre polynomials $f_n^c(x, r)$ (and the polynomials related to them) defined by

$$(1.14.3) \qquad (1-t)^{-c} \exp\left[-\left(\frac{r}{(1-t)}\right)^r xt\right] = \sum_{n=0}^{\infty} f_n^c(x, r)\, t^n .$$

Further Panda [144] generalized above polynomials through generating function

$$(1.14.4) \qquad (1-t)^{-c} G\left(\frac{xt^s}{(1-t)^r}\right) = \sum_{n=0}^{\infty} g_n^c(x, r, s)\, t^n ,$$

where c is an arbitrary parameter, r is any integer positive or negative, and s = 1, 2, 3,..., and

$$G(z) = \sum_{n=0}^{\infty} \gamma_n z^n , \quad (\gamma_0 \neq 0) .$$

Further Sinha [157] (Also see - Corrigendum due to Chandel [37]) studied special case of g_n^c(x, r), when $\gamma_n = \frac{1}{n!}$, $\gamma_0 = 1$.

For special interest Chandel and Bhargava ([52], [54]) studied an interesting special cae of (1.14.4) when $\gamma_n = \frac{(b)_n}{n!}$.

$$(1.14.5) \qquad (1-t)^{-c}\left[1-\frac{x\,t^s}{(1-t)^r}\right]^{-b} = \sum_{n=0}^{\infty} \Gamma_n^{(b,c)}(x,r,s)\;t^n$$

and introduced their associated polynomials. Chandel and Chandel [58] also introduced a new class of polynomials through their generating function

$$(1.14.6) \qquad (1-pt^q)^{-c}\;G\left(\frac{x\,t}{(1-p\,t^q)^r}\right) = \sum_{n=0}^{\infty} g_n^c(x,p,q,r)\;t^n$$

and discussed their related polynomials.

The generalization of all polynomials of Louville [133] Legendre [132], Tchebychef (see [182]), Gegenbaur [106], Humbert [116], Pincherle (as stated in [116] and Kinney [122] led Gould [110]) to define the poloynomials, through generating functions

$$(1.14.7) \qquad (C - mxt + yt^m)^p = \sum_{n=0}^{\infty} t^n\;P_n\;(m,x,y,p,C)\;,$$

where m is positive integer and other parameters are unrestricted in general.

Srivastava [166] considered the class of generalized Hermite polynomials defined by generating function

$$(1.14.8) \qquad \sum_{n=0}^{\infty} \gamma_n^{(m)}(x)\;\frac{t^n}{n!} = G(mxt - t^m)\,.$$

For its special case $G(z) = e^z$, see Chandel [30].

Chandel and Yadava [186] unified the study of above two classes (1.14.7) and (1.14.8) by considering the following generating function for certain polynomial systems :

$$(1.14.9) \qquad G\,(C - mxt + yt^q) = \sum_{n=0}^{\infty} g_n\;(m,x,y,q,C)\,t^n\,.$$

Inspired by (1.14.5) and (1.14.7), Chandel and Bhargava [55] introduced a class of polynomials through generating function

$$(1.14.10) \qquad [C - mxt + yt^m]^p\left[1-\frac{r^r\,x\,t^s}{(C - mxt + yt^m)^r}\right]^{-q} = \sum_{n=0}^{\infty} B_n^{(p,q)}\;(m,x,y,r,s,c)\,t^n\;,$$

where m, s are positive integers and other parameters are unrestricted in general. They also studied their related polynomials.

Further, to unify the study of four general classes (1.14.4), (1.14.6), (1.14.7) and (1.14.10) Chandel [38] introduced a class of polynomials through the generating function.

$$(1.14.11)\qquad (C - mxt + yt^m)^p\ G\left[\frac{r^r x\, t^s}{(C - mxt + yt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} R_n^p\ (m, x, y, r, s, C)\, t^n\ ,$$

and also discussed its special case when $\gamma_n = \dfrac{(-1)^n}{n!}$.

Chandel and Dwivedi ([64], [65]) also considered polynomial system through generating function

$$(C - mxt + yt^m)^p\ G\left[\frac{r^r z\, ts^s}{(C - mxt + yt^m)^r}\right]$$

and

$$(C - mxt + yt^m)^p\ =\ G\left[\frac{r^r x\, t^s}{(C - mxt + yt^m)^r}\right];$$

and discussed their special cases and related polynomials.

To further generalize (1.14.9), Chandel and Yadava [87] introduced some polynomial system of several variables by means of generating function

$$(1.14.12)\qquad G\,(a_0 + a_1x_1t + \ldots + a_mx_mt^m) = \sum_{n=0}^{\infty} A_{n,m}^{a_0,\ldots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} t^n$$

and discussed their special cases.

To further generalize (1.14.11) and the polynomials of Chandel and Dwivedi ([64], [65]), Chandel and Yadava [87] introduced a polynomial system of several variables through generating function

$$(1.14.13)\ (a_0 + A_1X_1t + \ldots + a_mx_mt^m)^p\ G\left[\frac{r^r x_i\, t^s}{(a_0 + a_1x_1t + \ldots + a_mx_mt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} B_{n,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1\ ,\ x_i \\ \vdots \\ x_m \end{bmatrix} t^n$$

and discussed their special cases.

Recently Chandel, Agrawal and Kumar [44] introduced a multivariable analogue of Gould - Hopper's polynomials [109], defined by generating function

$$(1.14.14)\qquad \sum_{m_1,\ldots,m_n=0}^{\infty} H_{m_1,\ldots,m_n}^{(h,\, m,\, v,\, p)}\ (x_1,\ldots,x_n)\ \frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}$$

$$=\ \exp\ [h\,(t_1^m + \ldots + t_n^m)]\,.\,[1 + v(x_1t_1 + \ldots + x_nt_n)]^p$$

and discussed its generalization through generating function

$$\text{(1.14.15)} \qquad \exp\left\{h\,(t_1^{m}+\ldots+t_n^{m})\right\} \cdot G\,[v(x_1t_1+\ldots+x_nt_n)]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} S_{m_1,\ldots,m_n}^{(h,\,m,\,v)}\,(x_1,\ldots,x_n)\,\frac{t_1^{m_1}}{m_1\,!}\cdots\frac{t_n^{m_n}}{m_n\,!}$$

Recently Chandel and Sahgal [77] introduced a multivariable analogue of Panda's polynomials [144], through generating function

$$\text{(1.14.16)} \qquad (1-t_1)^{-c_1}\ldots(1-t_m)^{-c_m}\left[1-\frac{x_1\,t_1^{s_1}}{(1-t_1)^{r_1}}\cdots\frac{x_m\,t_m^{s_m}}{(1-t_m)^{r_m}}\right]^{-b}$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} \Gamma_{n_1,\ldots,n_m}^{(b\,;\,c_1,\ldots,c_m\,;\,r_1,\ldots,r_m\,;\,s_1,\ldots,s_m}\,(x_1,\ldots,x_m)\;t_1^{n_1}\ldots t_m^{n_m}\,,$$

where b, $c_1,\ldots,c_m$ are any parameters, $r_1,\ldots,r_m$ are any integers positive or negative while $s_1,\ldots,s_m$ are positive integers.

They also considered its generalization through generating function

$$\text{(1.14.17)} \qquad (1-t_1)^{-c_1}\ldots(1-t_m)^{-c_m}\,G\left(\frac{x_1\,t_1^{s_1}}{(1-t_1)^{r_1}}\cdots\frac{x_m\,t_m^{s_m}}{(1-t_m)^{r_m}}\right)$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(c_1,\ldots,c_m\,;\,r_1,\ldots,r_m\,;\,s_1,\ldots,s_m)}\,(x_1,\ldots,x_m)\;t_1^{n_1}\ldots t_m^{n_m}$$

and discussed other special cases.

Very recently, Chandel and Sahgal [78] introduced a multivariable analogue of Gould - Hopper's polynomials [109] and Gould polynomials [110] through generating relation

$$\text{(1.14.18)} \qquad \sum_{n_1,\ldots,n_m=0}^{\infty} P_{n_1,\ldots,n_r}^{(m_1,\ldots,m_r\,;\,M_1,\ldots,M_r\,;\,h_1,\ldots,h_r\,;\,p)}\,(x_1,\ldots,x_r)\,\frac{t_1^{n_1}}{n_1\,!}\cdots\frac{t_r^{n_r}}{n_r\,!}$$

$$= \left(1+m_1x_1t_1+h_1t_1^{M_1}+\ldots+m_rx_rt_r+h_r\,t_r^{M_r}\right)^{p}\,,$$

where $M_1,\ldots,M_r$ are positive integers and $m_1,\ldots,m_r$, $h_1,\ldots,h_r$ are any numbers real or complex independent of variables $x_1,\ldots,x_r$. They also gave following generalization of (1.14.18) through generating relation

$$\text{(1.14.19)} \qquad \sum_{n_1,\ldots,n_r} g_{n_1,\ldots,n_r}^{(m_1,\ldots,m_r\,;\,M_1,\ldots,M_r\,;\,h_1,\ldots,h_r\,)}\,(x_1,\ldots,x_r)\,\frac{t_1^{n_1}}{n_1\,!}\cdots\frac{t_n^{n_r}}{n_r\,!}$$

$$= G\,\left(m_1\,x_1\,t_1+h_1\,t_1^{M_1}+\ldots+m_r\,x_r\,t_r+h_r\,t_r^{M_r}\right)\,.$$

In this thesis we shall extend the above work and introduce and study new multivariable analogue of Gould and Hopper's polynomials through their generating function (Chapter V).

In chapter VI, we shall discuss generating functions involving hypergeometric function of four variables (Chandel and Tiwari [79]). In chapter VII, we shall discuss multilinear generating functions.

(II) BY DEFINING SUITABLE RODRIGUE'S FORMULA.

The classical orthogonal polynomials have a generalized Rodrigues' formula (1.6.4). Following this formula, several mathematicians defined their polynomials through Rodrigues' formulae.

Krall and Frink [125] introduced a class of polynomials called "Bessel Polynomials" through Rodrigues' formula

(1.14.20) $$y_n(x; a, b) = b^{-n} x^{2-a} e^{b/x} D^n \left[x^{m+a-2} e^{-b/x}\right]$$

Agrawal [2] showed that Bessel polynomials are limiting cases of Jacobi polynomials. In a way to generalize Laguerre and Hermite polynomials, Gould and Hopper [109] introduced a function through Rodrigues' formula

(1.14.21) $$H_n^r(x, a, p) = (-1)^n x^{-\alpha} e^{px^r} \frac{d^n}{dx^n} (x^\alpha e^{-px^r}),$$

and to generalize Laguerre and Humbert polynomials, Singh and Srivastava [158] (also see Chatterjea [20]) defined the polynomials by Rodrigues' formula

(1.14.22) $$L_n^{(\alpha)}(x, r, p) = \frac{x^{-\alpha} e^{px^r}}{n!} D^n (x^{\alpha+n} e^{-px^r}).$$

Chatterjea [22], and Kharaize [124] gave the generalizations of Hermite polynomials. Chatterjea [19] studied generalized Bessels polynomials. Further Chatterjea [21] has defined a generalized function by the Rodrigues' formula

(1.14.23) $$F_n^{(r)}(x; a, k, p) = x^{-a} e^{px^r} D^n (x^{kn+a} e^{-px^r}).$$

It includes, Hermite, Laguerre, Bessel polynomials and the generalized Hermite function of Gould and Hopper [109] as special cases.

Riordon [148] considered the polynomials through Rodrigues' formula

(1.14.24) $$H_n[g, h] = (-1)^n e^{-hg} D e^{hg},$$

where h is constant and g is some specified function of x. Srivastava ans Singhal [178] introduced a class of polynomials defined by a generalized Rodrigues' formula. Further Srivastava and Panda [177] unified the several Rodrigues' formulas to define a general sequence of functions. Chandel and Agrawal [40] studied generalized Jacobi polynomials defined by Rodrigues' formula

(1.14.25)
$$P_n^{(\alpha,\beta)}(x; p, q, r, s, c, d) = \frac{(x^r+c)^{-\alpha}(x^s+d)^{-\beta}}{2^n n!} D^n \left[(x^r+c)^{np+\alpha} (x^s+d)^{nq+\beta}\right],$$

To generalize the Rodrigues' formula, some workers replaced operator D by xD or x^kD or $x^k(a + xD)$.

Few examples are given below :

Toscano [184] studied in detail the polynomials defined by

(1.14.26) $$G_n^{(\alpha)}(x) = x^{-\alpha} e^x (xD)^n \{x^\alpha e^{-x}\}.$$

Singh [159] introduced generalized Truesdell polynomials through Rodrigues' formula

(1.14.27) $$T_n^{(\alpha)}(x, r, p) = x^{-\alpha} e^{px^r} (xD)^n \{x^\alpha e^{-px^r}\}$$

while Shrivastava [181] considered some more general Truesdell polynomials defined by

(1.14.28) $$G_n(h, g) = e^{-hg} (xD)^n e^{hg},$$

where h is constant and g is function of x.

Chak [17] defined a function $G_{n,k}^{(\alpha)}(x)$ by

(1.14.29) $$G_{n,k}^{(\alpha)}(x) = x^{-\alpha-nk+n}\, e^{x}\, (x^k D)^n\, e^{-x}\, x^{\alpha}.$$

These are in more resemblance with the functions defined by Srivastava [163] through

(1.14.30) $$L_{n,\lambda}^{(\gamma)}(x) = \lambda^n\, x^{-(\gamma+n+1)/\lambda}\, (x^{1+1/\lambda} D)^n \left(e^{-x}\, x^{\gamma+1/\lambda}\right).$$

Following Singh [159] and Chak [17], Chandel ([32], [33]) introduced a generalized class of polynomials defined by

(1.14.31) $$T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha}\, e^{px^r}\, (x^k D)^n \left\{ x^{\alpha}\, e^{-px^r} \right\}.$$

$T_n^{(\alpha,k)}(x,r,p)$ are also generalized Stirling polynomials. Therefore Chandel ([33], [36]) also studied generalized Stirling numbers and polynomials. For Stirling numbers and polynomials also see Chandel-Yadava [85] and Chandel and Dwivedi [59].

Motivated by (1.14.28) and (1.14.31) Chandel [34], further introduced a generalized class of polynomials through

(1.14.32) $$G_n(h,g,k) = e^{-hg(x)}\, \Omega_x^n \left\{ e^{hg(x)} \right\},$$

where $\Omega_x = x^k \dfrac{d}{dx}$, h is constant and $g_{(x)}$ is differentiable function of x. It was an interesting unification, since these polynomials may also be regarded as the generalization of Laguerre, Hermite, Bessel, Truesdell, Bell polynomials and the polynomials of Riordan Chatterjea, Chak, Gould - Hopper, Chandel, Shrivastava, Srivastava-Singh and Singh etc.

For special interest Chandel and Agrawal [41] also studied the polynomials defined by

(1.14.33) $$T_{n,e,f,g}^{(\alpha,\beta,\gamma,k)}(x;a,b,c,d,p,r) = \frac{(ax+b)^{-\alpha}\, (cx+d)^{-\beta}\, x^{-\gamma}\, e^{px^r}}{n!}$$
$$\Omega_x^n \left[(ax+b)^{(\alpha+en)}\, (cx+d)^{\beta+fn}\, x^{\gamma+g_n}\, e^{-px^r} \right].$$

where a, b, c, d, e, f, g, p, r, α, β, γ, $k(\neq 1)$ are arbitrary numbers independent of x.

Joshi and Prajapat [119] generalized certain classical polynomials through Rodrigues' formula

(1.14.34) $$M_{v_n}^{(\alpha)}(x,a,k) = \frac{1}{n!}\, x^{-\alpha-nk}\, e^{p_v(x)}\, T_{a,k}^n \left\{ x^{\alpha}\, e^{-p_v(x)} \right\},$$

where $T_{a,k} = x^k (a + xD)$.

For the use of this operator $T_{a,k}$ also see Patial and Thakare ([141], [142], [143]). Chandel and Bhargava [53] unified the study of three classes (1.14.32), (1.14.33) and (1.14.34) by introducing a sequence of functions through

(1.14.35) $$G_n(a,k;h,g(x)) = e^{-hg(x)}\, T_{a,k}^n (e^{hg(x)}),$$

Further Bhargava [12] introduced general sequence of functions defined by

(1.14.36) $$G(a,k,p;g(x),h(x)) = e^{-pg(x)}\, T_{a,k}^n \left\{ (h(x))^n\, e^{pg(x)} \right\}.$$

Recently Chandel and Agrawal [49] introduced a generalization of (1.14.32), through

(1.14.37) $$S_n^{(\alpha, k)}(h, g) = [1 - h\, g(x)]^{\alpha}\, \Omega_x^n \left\{(1 - h\, g(x))^{-\alpha}\right\},$$

where α, h, k are independent of x.

Further Chandel and Agrawal [50] gave an unified presentation of general sequence of functions defined by Rodrigues' formula

(1.14.38) $$R_n^{(a, b, k, p)}(h(x), g(x)) = [1 - pg(x)]^{-b}\; T_{a, k}^n \left\{h(x)^n\, (1 - pg(x)^{-b})\right\} .$$

In the present thesis, we shall introduce a multivariable analogue of Hermite polynomials (Chandel and Tiwari [81]) defined by a Rodrigues' formula (Chapter II). We shall further introduce a multivariable analogue of Gould and Hopper's polynomials (Chandel and Tiwari [80]) defined by a Rodrigues' formula. We shall also discuss its generalization (Chapter III, IV).

1.15 APPLICATIONS OF SPECIAL FUNCTIONS.

For applications of Special Functions in mixed boundary value problems one may refer to Sneddon [156], Chandel [39] discussed a mixed boundary value problem on heat conduction and determined the temprature at any point on the surface of sphere by solving dual series equations involving the Legendre polynomials, Chandel - Bhargava [57], Chandel - Dwivedi [66] and Chandel - Yadava [89] discussed a problem on heat conduction employing generalized Kampé de Fériet function of Srivastava-Daoust [170], Srivastava's hypergeometric function of three variables [167], and multiple hypergeometric functions of Srivastava-Daoust [171] (defined by (1.12.1) and (1.12.2)), respectively.

Chandel and Bhargava [56] discussed a problem on cooling of a heated cylinder using generalized Kampé de Fériet function of Srivastava and Daoust [170]. Chandel and Gupta [73] used multiple hypergeometric function of Srivastava and Daoust [171] in the solution of a problem on heat conduction in a finite bar, while Chandel and Gupta ([69], [71]) made application of multivariable H-function of Srivastava and Panda defined by (1.13.1) in the problems of heat conduction and in cooling of a heated cylinder, respectively. Chandel, Agrawal and Kumar [43] used multivariable H-function of Srivastava and Panda in a problem on electrostatic potential in spherical regions. Furtehr Chandel, Agrawal and Kumar [45] evaluated an integral involving Kampé de Fériet function and multivariable H-function of Srivastava and Panda, and then applied it to solve a problem on a circular disk. Chandel, Agarwal and Kumar [47] also used multivariable H- function of Srivastava and Panda in Fourier series.

Further Chandel, Agarwal and Kumar [48] made application of Lauricella's $F_D^{(n)}$ in determining velocity coefficient of chemical reaction.

In the present thesis in chapter VIII, we shall make application of multiple hypergeometric function of Srivastava and Daoust [171] in mixed boundary value problem involving Laplace equation.

Further in Chapter IX, we shall make appeal to multiple hypergeometric function of Srivastava and Daoust in two different boundary value problems.

Finally in Chapter X, we shall make an appeal to multiple hypergeometric function of Srivastava and Daoust [171] and multivariable H-function of Srivastava and Panda defined by (1.13.1), in solving a potential problem on a circular disk.

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CHAPTER II

A MULTIVARIABLE ANALOGUE OF HERMITE POLYNOMIALS DEFINED BY RODRIGUES' FORMULA

2.1 <u>INTRODUCTION.</u> Recently Beniwal and Saran [1] have studied two variable analogue $L_{m,n}^{(a,b,c)}(x, y)$ of Laguerre polynomials associated with Appell function F_2 defined by

$$(2.1.1) \qquad \mathbf{L}_{m,n}^{(a,b,c)}(x, y, z) = \frac{(b)_n (c)_n}{m!\, n!} F_2 [a, -m, -n; b, c; x, y],$$

from which it is clear that

$$(2.1.2) \qquad \lim_{a \to \infty} \mathbf{L}_{m,n}^{(a,b,c)}\left(\frac{x}{a}, \frac{y}{a}\right) = L_m^{(b-1)}(x)\, L_n^{(c-1)}(y) .$$

Motivated by above work, very recently Raizada and Shrivastava [4] have defined two variable analogue $P_{k,n}^{(v,)}(x, y)$ of Legendre polynomials by the integral

$$(2.1.3) \qquad \mathbf{P}_{k,n}^{(v)}(x, y) = \frac{2^2}{n!\, k!\, \pi} \int_o^\infty \int_0^\infty [\exp - (t^2 + T^2)]\, t^k T^n H_{k,n}^{(v)}(xt, yt)\, dt\, dT ,$$

where $P_{k,n}^{(v)}(x, y)$ is two variable analogue of Hermite polynomials defined by Raizada and Shrivastava [5] in the following way :

$$(2.1.4) \qquad \sum_{n,k=0}^{\infty} \frac{H_{k,n}^{(v)}(x, y)}{k!\, n!} t^k T^n = \exp [- (t^2 + T^2)\, (1 + 2xt + 2yt)^{(v)} .$$

From (2.1.3) it is clear that

$$(2.1.5) \qquad \lim_{v \to \infty} \mathbf{P}_{k,n}^{(v)}\left(\frac{x}{v}, \frac{y}{v}\right) = P_k(x)\, P_n(y)$$

while from (2.1.4), it is clear that

$$(2.1.6) \qquad \lim_{v \to \infty} \mathbf{H}_{k,n}^{(v)}\left(\frac{x}{v}, \frac{y}{v}\right) = H_k(x)\, H_n(y)$$

where $P_n(x)$ and $H_n(x)$ are Legendre polynomials and Hermite polynomials respectively.

Motivated by the above work, in this chapter we introduce the multivariable analogue $H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)$ of Hermite polynomials, defined by Rodrigues' formula

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A paper from this chapter, entitled "<u>A multivariable analogue of Hermite polynomials</u>" has been published in Ganita Sandesh, 5(1991), 92-95.

$$\text{(2.1.7)} \qquad \mathbf{H}_{n_1,\ldots,n_m}^{(b\,;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

$$= (-1)^{n_1,\ldots,n_m}\left(1+h_1x_1^{r_1}+\ldots+h_mx_m^{r_m}\right)^{b}$$

$$\frac{d^{n_1}}{d\,x_1^{n_1}}\cdots\frac{d^{n_m}}{d\,x_m^{n_m}}\left(1+\,h_1\,x_1^{r_1}+\ldots+\,h_m\,x_m^{r_m}\right)^{-b}$$

where $n_1,\ldots,n_m$ are positive integers while $h_1,\ldots,h_m;\, r_1,\ldots,r_m$ and b are any numbers real or complex.

From (2.1.7) we have

$$\text{(2.1.8)} \qquad \lim_{b\to\infty} \mathbf{H}_{n_1,\ldots,n_m}^{(b,\frac{1}{b},\ldots,\frac{1}{b};\,2,\ldots,2)}(x_1,\ldots,x_m)$$

$$= H_{n_1}(x_1)\,\cdots\,H_{n_m}(x_m)\ ,$$

where $H_n(x)$ are Hermite polynomials.

2..2 GENERATING RELATION.

Replacing x_i by $\frac{1}{x_i}$, $i = 1,\ldots, m$ in (2.1.7) and applying the result due to Chandel and Agrawal ([2], p.88(3.2)) (Also see earlier reference due to Edwards [3, p.506, Misc, Ex No. 15]).

$$\text{(2.2.1)} \qquad e^{t\,\Omega x}\left\{f(x)\right\} = f\left(\frac{x}{1-xt}\right),\ \Omega_x = x^2\frac{\partial}{\partial x},$$

we derive generating relation

$$\text{(2.2.2)} \qquad \sum_{n_1,\ldots,n_m=0}^{\infty}\mathbf{H}_{n_1,\ldots,n_m}^{(b;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)\ \frac{t_1^{\,n_1}}{n_1\,!}\ \cdots\cdots\ \frac{t_m^{\,n_m}}{n_m\,!}$$

$$= \left[1+h_1\,x_1^{r_1}+\ldots+h_m\,x_m^{r_m}\right]^{+b}\left[1+h_1\,(x_1-t_1)^{r_1}+\ldots+h_m\,(x_m-t_m)^{r_m}\right]^{-b}.$$

2.3 APPLICATION OF GENERATING RELATION.

Making an appeal to generating relation (2.2.2), we obtain

$$\text{(2.3.1)} \qquad \mathbf{H}_{n_1,\ldots,n_m}^{(b+b'\,;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

$$= \sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}\ H_{n_1-k_1,\ldots,n_m-k_m}^{(b;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

$$\mathbf{H}_{k_1,\ldots,k_m}^{(b'\,;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

Differentiating generating relation (2.2.2) w.r.t. t_1 and equating the coefficients of $t_1^{n_1}\,\ldots\,t_m^{n_m}$ both the sides, we get

$$\text{(2.3.2)} \qquad \left(1+h_1\,x_1^{r_1}+\ldots+h_m\,x_m^{r_m}\right)\mathbf{H}_{n_1+1,\,n_2,\ldots,n_m}^{(b\,;\,h_1,\ldots,h_m\,;\,r_1,\ldots,r_m)}(x_1,\ldots,x_m)$$

$$= b\, h_1\, r_1\, x_1^{r_1 - 1} \sum_{k=0}^{\min(r_1 - 1,\, n_1)} \binom{r_1 - 1}{k} \left(\frac{-1}{x_1}\right)^k \frac{n_1!}{(n_1 - k)!}$$

$$\mathbf{H}^{(b+1,\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1 - k,\, n_2, \ldots, n_m} (x_1, \ldots, x_m),$$

which can be generalized further in the form :

$$(2.3.3) \quad \left(1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}\right) \mathbf{H}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1},\, n_i + 1,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

$$= \; b\, h_i\, r_i\, (x_i)^{r_i - 1} \sum_{k=0}^{\min(r_i - 1,\, n_i)} \binom{r_i - 1}{k} \left(\frac{-1}{x_i}\right)^k \frac{n_i!}{(n_i - k)!}$$

$$\mathbf{H}^{(b+1;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1},\, n_i - k,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

where $i = 1, \ldots, m$.

Now differentiating generating relation (2.2.2) partially w.r.t. x_1 and equating the coefficients of $t_1^{n_1} \ldots t_m^{n_m}$ both the sides, we establish

$$(2.3.4) \quad \left[b\, r_1\, h_1\, x_1^{r_1 - 1} - \left(1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}\right) \frac{\partial}{\partial x_1}\right]$$

$$\mathbf{H}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)$$

$$= b\, r_1\, h_1\, x_1^{r_1 - 1} \sum_{k=0}^{\min(r_1 - 1,\, n_1} \binom{r_1 - 1}{k} \left(-\frac{1}{x_1}\right)^k \frac{n_1!}{(n_1 - k)!}$$

$$\mathbf{H}^{(b+1;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)$$

which can be generalized in the following form :

$$(2.3.5) \quad \left[b\, r_i\, h_i\, (x_i)^{r_i - 1} - \left(1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}\right) \frac{\partial}{\partial x_i}\right]$$

$$\mathbf{H}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)$$

$$= b\, r_i\, h_i\, x_i^{r_i - 1} \sum_{k=0}^{\min(r_i - 1,\, n_i)} \binom{r_i - 1}{k} \left(-\frac{1}{x_i}\right)^k \frac{n_i!}{(n_i - k)!}.$$

$$\mathbf{H}^{(b+1,;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1},\, n_i + 1,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

combining (2.3.3) and (2.3.5) we further derive

$$\text{(2.3.6)} \quad \left[b\, r_i\, h_i\, x_i^{r_i - 1} - \left(1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}\right) \frac{\partial}{\partial x_i}\right]$$

$$\mathbf{H}^{(b;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)$$

$$= \left(1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}\right) \mathbf{H}^{(b;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_i, \ldots, n_{i-1} n_i + 1\, ,\, n_{i+1}\, , \ldots, n_m} (x_1, \ldots, x_m)$$

where $i = 1, \ldots, m$.

From (2.3.6)

$$\text{(2.3.7)} \left(\frac{b\, r_i\, h_i\, x_i^{r_i}}{1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}} - \frac{\partial}{\partial x_i}\right) \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)$$

$$= \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1}\, ,\, n_i + 1\, ,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

For briefly we take $\dfrac{b\, r_i\, h_i\, x_i^{r_i - 1}}{1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}} - \dfrac{\partial}{\partial x_i} = \mathfrak{S}_i$

Therefore

$$\text{(2.3.8)} \quad \mathfrak{S}_i \left\{\mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)\right\}$$

$$= \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1}\, ,\, n_i + 1,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

$$\text{(2.3.9)} \quad \mathfrak{S}_i^j \left\{\mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, r_m)\right\}$$

$$= \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1}\, ,\, n_i + j\, ,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m)$$

which gives

$$\text{(2.3.10)} \quad e^{t\, \mathfrak{S}_i} \left\{\mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)\right\}$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j\,!} \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_{i-1} + n_i + n_j\, ,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m).$$

where $i = 1, \ldots, m$

specially for $j = n_j$ in (2.3.9), we have

$$\text{(2.3.11)} \quad \mathfrak{S}_i^{n_j} \left\{\mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_m} (x_1, \ldots, x_m)\right\}$$

$$= \mathbf{H}^{(b\, ;\, h_1, \ldots, h_m\, ;\, r_1, \ldots, r_m)}_{n_1, \ldots, n_i + n_j\, ,\, n_{i+1}, \ldots, n_m} (x_1, \ldots, x_m).$$

Also

$$(2.3.12)\qquad \prod_{i=1}^{m} \mathfrak{D}_i^{k_i} \left\{ \mathbf{H}_{n_1,\dots,n_m}^{(b\,;\,h_1,\dots,h_m\,;\,r_1,\dots,r_m)} (x_1,\dots,x_m) \right\}$$

$$= \mathbf{H}_{n_1+k_1,\dots,n_m+k_m}^{(b\,;\,h_1,\dots,h_m\,;\,r_1,\dots,r_m)} (x_1,\dots,x_m)$$

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CHAPTER III

MULTIVARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS DEFINED BY RODRIGUES' FORMULA

3.1 INTRODUCTION

In the previous chapter II we studied a multivariable analogue of Hermite polynomials defined by Rodrigues' formula (2.1.7). Recently Chandel and Sahgal [2,3] have studied multivariable analogue of Panda's polynomials, and Gould's and Gould's- Hopper's polynomials respectively through their generating function s. Motivated by the above works, in this chapter, we introduce and study the multivariable analogue

$$\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m\,;\,b)}(x_1,\ldots,x_m)$$

of Gould's and Hopper's polynomials [5], through Rodrigues' formula

(3.1.1)
$$\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,b)}(x_1,\ldots,x_m)$$
$$= (-1)^{n_1,\ldots,n_m}\, x_1^{-a_1}\ldots x_m^{-a_m}\left[1+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right]^b$$
$$\frac{d^{n_1}}{dx_1^{n_1}}\ldots\frac{d^{n_m}}{dx_m^{n_m}}\left\{x_1^{a_1}\ldots x_m^{a_m}\left(1+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right)^{-b}\right\},$$

where parameters $r_1,\ldots,r_m, a_1,\ldots,a_m, p_1,\ldots,p_m$, b are unrestricted in general but independent of variables $x_1,\ldots,x_m$.

It is clear that for $a_1 = \ldots = a_m = 0$, (3.1.1) reduces to (2.1.7)

and

(3.1.2)
$$\lim_{b\to\infty}\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1/b,\ldots,p_m/b\,;\,b)}(x_1,\ldots,x_m)$$
$$= \mathbf{H}_{n_1}^{r_1}(x_1, a_1, p_1)\ldots\mathbf{H}_{n_m}^{r_m}(x_m, a_m, p_m)$$

where $H_n^r(x, a, p)$ are Gould and Hopper's polynomials defined by Rodrigues' formula [5] (Also see Srivastava and Manocha [8, p.77 eq.(12)])

(3.1.3)
$$\mathbf{H}_n^r(x, a, p) = (-1)^n x^{-a}\, e^{px^r}\frac{d^n}{dx^n}\left\{x^a e^{-px^r}\right\}$$

3.2 GENERATING FUNCTION.

Replacing each x_i by $1/x_i$, i = 1,..., m, we derive from (3.1.1)

$$(3.2.1)\ \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)} (1/x_1,\ldots,1/x_m)\ \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$x_1^{a_1} \cdots x_m^{a_m} \left[1 + p_1 x_1^{-r_1} + \ldots + p_m x_m^{-r_m}\right]^b$$

$$e^{t_1 \Omega_{x_1} + \ldots + t_m \Omega_{x_m}} \left\{ x_1^{-a_1} \cdots x_m^{-a_m} \left(1 + p_1 x_1^{-r_1} + \ldots + p_m x_m^{-r_m}\right)^{-b} \right\}$$

which by making an appeal to the result due to Chandel and Agrawal [1,p.88 (3.2)] (Also see earlier reference due to Edwards [4, p.506 Misc. Ex. No.15])

$$e^{t\Omega_x} f(x) = f\left(\frac{x}{1-xt}\right), \quad \Omega_x = x^2 \frac{\partial}{\partial x},$$

finally gives the generating relation

$$(3.2.2)\qquad \sum_{n_1,\ldots,n_m=0}^{\infty} \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)} (x_1,\ldots,x_m)\ \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= \left(1 - \frac{t_1}{x_1}\right)^{a_1} \cdots \left(1 - \frac{t_m}{x_m}\right)^{a_m} \left[\frac{1 + p_1 (x_1 - t_1)^{r_1} + \ldots + p_m (x_m - t_m)^{r_m}}{1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}}\right]^{-b}$$

3.3 EXPLICIT FORM

Starting with the generating relation (3.2.2), we obtain the following explicit form :

$$(3.3.1)\qquad \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)} (x_1,\ldots,x_m)$$

$$= \left[1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right]^b (-a_1)_{n_1} \cdots (-a_m)_{n_m} \left(\frac{1}{x_1}\right)^{n_1} \cdots \left(\frac{1}{x_m}\right)^{n_m}$$

$$\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(b)_{k_1+\ldots+k_m} (a_1+1)_{r_1 k_1} \cdots (a_m+1)_{r_m k_m}}{(a_1+1-n_1)_{r_1 k_1} \cdots (a_m+1-n_m)_{r_m k_m}} \frac{(-p_1 x_1^{r_1})^{k_1}}{k_1!} \cdots \frac{(-p_m x_m^{r_m})^{k_m}}{k_m!}$$

which can be written in the form of generalized multiple hypergeometric function of Srivastava and Daoust [6, 7] see also Srivastava and Karlsson [9, p.21, eq.(21)]) :

$$(3.3.2)\qquad \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)} (x_1,\ldots,x_m)$$

$$= \left[1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right]^b (-a_1)_{n_1},\ldots,(-a_m)_{n_m} \frac{1}{x_1^{n_1}} \cdots \frac{1}{x_m^{n_m}}$$

$$F^{1:1;\ldots;1}_{0:1;\ldots;1} \left(\begin{matrix} [b:1,\ldots,1] : [a_1+1:r_1];\ldots;[a_m+1:r_m]; \\ - : [a_1+1-n_1:r_1];\ldots;[a_m+1-n_m:r_m]; \end{matrix} \ -p_1 x_1^{r_1},\ldots, p_m x_m^{r_m} \right).$$

3.4 APPLICATIONS OF GENERATING RELATION

Making an appeal to generating relation (3.2.2), we obtain

$$(3.4.1)\qquad \mathbf{H}_{n_1,\ldots,n_m}^{(a_1+c_1,\ldots,a_m+c_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b+d)} (x_1,\ldots,x_m)$$

$$= \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} \mathbf{H}_{n_1-k_1,\dots,n_m-k_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m)$$

$$\mathbf{H}_{k_1,\dots,k_m}^{(c_1,\dots,c_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m).$$

Differentiating generating relation (3.2.2) partially with respect to t_1 and equating coefficients of $t_1^{n_1},\dots,t_m^{n_m}$ both the sides, we derive recurrence relation

$$(3.4.2) \qquad x_1 \mathbf{H}_{n_1+1,\,n_2,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m)$$

$$= (n_1 - a_1)\,\mathbf{H}_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m)$$

$$+ \frac{b\, r_1 p_1 x_1^{r_1}}{1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m}} \mathbf{H}_{n_1,\dots,n_m}^{(a_1+r_1-1,\,a_2,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b+1)}(x_1,\dots,x_m)$$

$$- \frac{b\, n_1 r_1 p_1 x_1^{r_i-1}}{1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m}} \mathbf{H}_{n_1,\dots,n_m}^{(a_1+r_1-1,\,a_2,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b+1)}(x_1,\dots,x_m)$$

which can be further generalized in the form

$$(3.4.3) \qquad x_i \mathbf{H}_{n_1,\dots,n_{i-1},\,n_i+1,\,n_{i+1},\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m)$$

$$= (n_i - a_i)\, H_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b)}(x_1,\dots,x_m)$$

$$+ \frac{b\, r_i p_i x_i^{r_i}}{1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m}}$$

$$\mathbf{H}_{n_1,\dots,n_m}^{(a_1,\dots,a_{i-1},\,a_i+r_i-1,\,a_{i+1},\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b+1)}(x_1,\dots,x_m)$$

$$- \frac{b\, n_i r_i p_i x_i^{r_i}}{1 + p_1 x_1^{r_1} + \dots + p_m x_m^{r_m}}$$

$$\mathbf{H}_{n_1,\dots,n_{i-1},\,n_i-1,\,n_{i+1}\cdots n_m}^{(a_1,\dots,a_{i-1},\,a_i+r_i-1,\,a_{i+1},\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m;\,b+1)}(x_1,\dots,x_m)$$

$i = 1,\dots, m.$

Making an appeal to generating relation (3.2.2), we also derive differential recurrence relation

$$(3.4.4)\qquad \left[b\,p_1 r_1 x_1^{r_1+1} - \left(1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right) x_1^2 \frac{\partial}{\partial x_1}\right]$$

$$\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$= b\,p_1 r_1 x_1^{r_1+1}\,\mathbf{H}_{n_1,\ldots,n_m}^{(a_1+r_1-1,\,a_2,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b+1)}(x_1,\ldots,x_m)$$

$$- a_1 n_1\left[1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right]\mathbf{H}_{n_1-1,\,n_2,\ldots,n_m}^{(a_1-1,\,a_2,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m),$$

which can be further generalized in the form

$$(3.4.5)\qquad \left[b\,p_i r_i x_i^{r_i+1} - \left(1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right) x_i^2 \frac{\partial}{\partial x_i}\right]$$

$$\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$=$$

$$b\,p_i r_i x_i^{r_i+1}\,\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i+r_i-1,\,a_{i+1},\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b+1)}(x_1,\ldots,x_m),$$

$$- a_i n_i\left[1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right]$$

$$\mathbf{H}_{n_1,\ldots,n_{i-1},\,n_i-1,\,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i-1,\,a_{i+1},\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m),$$

$i = 1,\ldots, m$.

We also derive a result

$$(3.4.6)\left(\frac{-b\,p_i r_i x_i^{r_i+1}}{1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}} + x_i^2 \frac{\partial}{\partial x_i}\right)\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$=$$

$$\frac{-n_i^2 a_i}{b + n_1 + \ldots + n_m - 1}\,\mathbf{H}_{n_1,\ldots,n_{i-1},\,n_i-1,\,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i-1,\,a_{i+1},\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$- a_i x_i\,\mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i-1,\,a_{i+1},\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$+ \frac{(b + n_1 + \ldots + n_m)}{n_i + 1} x_i^2\,\mathbf{H}_{n_1,\ldots,n_{i-1},\,n_i+1,\,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

A combination of (3.4.5) and (3.4.6) gives

$$(3.4.7)\ a_i n_i\left[1 + \frac{n_i}{b + n_1 + \ldots + n_m - 1}\right]$$

$$\mathbf{H}_{n_1,\ldots,n_{i-1},\,n_i-1,\,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_{i-1},\,a_i-1,\,a_{i+1},\ldots,a_m;\,r_1,\ldots,r_m;\,p_1,\ldots,p_m;\,b)}(x_1,\ldots,x_m)$$

$$= \frac{b\, p_i\, r_i\, x_i^{r_i + 1}}{1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}}$$

$$\mathbf{H}_{n_1, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i + r_i - 1, a_{i+1}, \ldots, a_m\, ;\, r_1, \ldots, r_m\, ;\, p_1, \ldots, p_m\, ;\, b+1)} (x_1, \ldots, x_m)$$

$$- a_i x_i\, \mathbf{H}_{n_1, \ldots, n_m}^{(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_m\, ;\, r_1, \ldots, r_m\, ;\, p_1, \ldots, p_m\, ;\, b)} (x_1, \ldots, x_m)$$

$$+ \frac{(b + n_1 + \ldots + n_m)}{n_i + 1} x_i^2\, \mathbf{H}_{n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_m}^{(a_1, \ldots, a_m\, ;\, r_1, \ldots, r_m\, ;\, p_1, \ldots, p_m\, ;\, b)} (x_1, \ldots, x_m).$$

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CHAPTER IV

GENERALIZED MULTIVARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS DEFINED BY RODRIGUES' FORMULA

4.1 INTRODUCTION.

Recently Chandel and Sahgal [2, 3] have studied multivariable analogue of Panda's polynomials, and Gould's and Gould-Hopper's polynomials through their generating function. Motivated by the above works and the work of chapter II, in chapter III, we introduced a multivariable analogue of Gould and Hopper's polynomials through their Rodrigues' formula (3.1.1). Now in this chapter, we further extend the work by introducing the generalized multivariable analogue

$$G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\, r_1,\dots,r_m;\, p_1,\dots,p_m)}(x_1,\dots,x_m)$$

of Gould and Hopper's polynomials [5] through Rodrigues' formula

$$(4.1.1)\qquad G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\, r_1,\dots,r_m;\, p_1,\dots,p_m)}(x_1,\dots,x_m)$$
$$= (-1)^{n_1+\dots+n_m} x_1^{-a_1}\dots x_m^{-a_m}\,[G(p_1 x_1^{r_1}+\dots+p_m x_m^{r_m})]^{-1}$$
$$\frac{d^{n_1}}{dx_1^{n_1}}\dots\frac{d^{n_m}}{dx_m^{n_m}}\left\{x_1^{a_1}\dots x_m^{a_m}\, G(p_1 x_1^{r_1}+\dots+p_m x_m^{r_m})\right\}$$

where parameters $r_1,\dots,r_m$; $a_1,\dots,a_m$; $p_1,\dots,p_m$ are unrestricted in general but independent of variables $x_1,\dots,x_m$ and

$$(4.1.2)\qquad G(z) = \sum_{n=0}^{\infty} \gamma_n z^n,\ \gamma_0 \neq 0.$$

4.2 GENERATING FUNCTION. Replacing each x_i by $1/x_i$, $i = 1,\dots, m$, we derive from (4.1.1)

$$(4.2.1)\qquad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\, r_1,\dots,r_m;\, p_1,\dots,p_m)}(1/x_1,\dots,1/x_m)\,\frac{t_1^{n_1}}{n_1!}\dots\frac{t_m^{n_m}}{n_m!}$$
$$= x_1^{a_1}\dots x_m^{a_m}\left[G\left(p_1 x_1^{-r_1}+\dots+p_m x_m^{-r_m}\right)\right]^{-1}$$
$$\exp(t_1\Omega_{x_1}+\dots+t_m\Omega_{x_m})\left\{x_1^{-a_1}\dots x_m^{-a_m}\, G\, p_1 x_1^{-r_1}+\dots+p_m x_m^{-r_m})\right\}$$

which by making an appeal to the result due to Chandel and Agarwal [1][p.88(3.2)] (Also see earlier reference due to Edwards [4][p.506 Misc.Ex. No. 15])

$$e^{t\,\Omega_x}\{f(x)\} = f\left(\frac{x}{1-xt}\right),\quad \Omega_x = x^2\frac{\partial}{\partial x}$$

A paper from this chapter, entitled "Multivariable analogue of Gould and Hooper's polynomials defined by Rodrigues' Formula" has been published in Indian J. Pure Appl. Math. 22 (1991) 757 - 761

finally gives the generating relation

$$\text{(4.2.2)}\qquad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left(1-\frac{t_1}{x_1}\right)^{a_1}\cdots\left(1-\frac{t_m}{x_m}\right)^{a_m}\frac{G\left[p_1(x_1-t_1)^{r_1}+\dots+p_m(x_m-t_m)^{r_m}\right]}{G\left(p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)}$$

4.3 Explicit Form

Starting with the generating relation (4.2.2), we derive the following explicit form :

$$\text{(4.3.1)}\qquad G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$=\frac{-(a_1)_{n_1}\cdots(-a_m)_{n_m}}{G\left(p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)}\;\frac{1}{x_1^{n_1}\cdots x_m^{n_m}}\sum_{n_1,\dots,n_m=0}^{\infty}\gamma_{k_1}+\dots+\gamma_{k_m}$$

$$\frac{(1+a_1)_{r_1k_1}\cdots(1+a_m)_{r_mk_m}}{(1+a_1-n_1)_{r_1k_1}\cdots(1+a_m-n_m)_{r_mk_m}}\;\frac{(p_1x_1^{r_1})^{k_1}}{k_1!}\cdots\frac{(p_mx_m^{r_m})^{k_m}}{k_m!}.$$

4.4 APPLICATION OF GENERATING RELATION

An appeal to generating relation (4.2.2) gives

$$\text{(4.4.1)}\qquad G_{n_1,\dots,n_m}^{(a_1+b_1,\dots,a_m+b_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$=\sum_{k_1=0}^{\min(n_1,[b_1])}\cdots\sum_{k_m=0}^{\min(n_m,[b_m])}\frac{(-b_1)_{k_1}\cdots(-b_m)_{k_m}}{x_1^{k_1}\cdots x_m^{k_m}}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}$$

$$G_{n_1-k_1,\dots,n_m-k_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m).$$

Again from generating relation (4.2.2), we derive the differential recurrence relation

$$\text{(4.4.2)}\qquad \left(\frac{x^2\frac{\partial}{\partial x_1}\left[g\left(p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)\right]}{G\left(p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)}+x_1^2\frac{\partial}{\partial x_1}\right)$$

$$G_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$=n_1a_1\,G_{n_1-1,\,n_2,\dots,n_m}^{(a_1-1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-a_1x_1\,G_{n_1,\dots,n_m}^{(a_1-1,a_2,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-x_1^2\,G_{n_1+1,\,n_2,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

which suggests them-results similar to above can be unified in the form

$$(4.4.3)\qquad \left(\frac{x_i^2 \frac{\partial}{\partial x_i} G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})}{G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})} + x_i^2 \frac{\partial}{\partial x_i} \right)$$

$$\mathbf{G}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m)} (x_1,\ldots,x_m)$$

$$= n_i a_i \, \mathbf{G}_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m)} (x_1,\ldots,x_m)$$

$$- a_i x_i \, \mathbf{G}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m)} (x_1,\ldots,x_m)$$

$$- x_i^2 \, \mathbf{G}_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m)} (x_1,\ldots,x_m)$$

$i = 1,\ldots, m.$

4.5 <u>SPECIAL CASES</u>. Particularly for $\gamma_n = \dfrac{(-1)^n (b)_n}{n!}$, (4.1.1) defines

$$(4.5.1)\qquad \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m\,;\,b)} (x_1,\ldots,x_m)$$

$$= (-1)^{n_1,\ldots,n_m} x_1^{-a_1} \ldots x_m^{-a_m} \left[1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right]^b$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \cdots \frac{d^{n_m}}{dx_m^{n_m}} \left\{ x_1^{a_1} \ldots x_m^{a_m} \left(1 + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right)^{-b} \right\}.$$

where parameters $r_1,\ldots,r_m, a_1,\ldots,a_m, p_1,\ldots,p_m, b$ are unrestricted in general but independent of variables $x_1,\ldots,x_m$.

For $\gamma_n = (-1)^n / n!$, (4.1.1) defines

$$(4.5.2)\qquad \mathbf{E}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1,\ldots,p_m)} (x_1,\ldots,x_m)$$

$$= (-1)^{n_1,\ldots,n_m} x_1^{-a_1} \ldots x_m^{-a_m} \exp\left\{ -\left(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right)\right\}$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \cdots \frac{d^{n_m}}{dx_m^{n_m}} \left\{ x_1^{a_1} \ldots x_m^{a_m} \exp\left(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right)\right\}$$

It is clear that

$$(4.5.3)\qquad \lim_{b\to\infty} \mathbf{H}_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m\,;\,r_1,\ldots,r_m\,;\,p_1/b,\ldots,p_m/b\,;\,b)} (x_1,\ldots,x_m)$$

$$\mathbf{H}_{n_1}^{r_1}(x_1, a_1, p_1) \ldots H_{n_m}^{r_m}(x_m, a_m, p_m)$$

and

(4.5.4) $$\mathbf{E}_{n_1,\dots,n_m}^{(a_1,\dots,a_m;\,r_1,\dots,r_m;\,p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= H_{n_1}^{r_1}(x_1, a_1, p_1)\ \dots\ H_{n_m}^{r_m}(x_m, a_m, p_m)$$

where $H_n^r(x, a, p)$ are Gould and Hopper's polynomials defined by Rodrigues' formula [5] (also see Srivastava and Manocha [6]. p.77 eq. (12)).

(4.5.5) $$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} \frac{d^n}{dx^n}\left\{x^a e^{-px^r}\right\}.$$

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CHAPTER - V

OTHER MULTIVARIABLE ANALOGUES OF GOULD AND HOPPER'S POLYNOMIALS DEFINED BY GENERATING RELATIONS

5.1 INTRODUCTION

In the chapter II, we introduce multivariable analogue of Hermite polynomials, defined by Rodrigues' formula (2.1.7) and in chapter III and IV, we introduce multivariable analogue of Gould and Hopper's polynomials [5](See also Srivastava and Manocha [7, p.86, eq.(27)]), through their Rodrigues' formulae (3.1.1) and (4.1.1) respectively.

In this chapter, we shall introduce two multivariable analogues of Gould and Hopper's polynomials through their generating relations and discuss their further generalizations and special cases.

5.2 FIRST MULTIVARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS. Recently Chandel, Agarwal and Kumar [2] introduce a multivariable analogue of Gould and Hopper's polynomials [5], defined by generating relation.

(5.2.1)
$$\sum_{m_1,\ldots,m_n=0}^{\infty} H^{(h,m,v,p)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}$$
$$= \exp\left[h\left(t_1^m+\ldots+t_n^m\right)\right]\cdot\left[1+v\left(x_1t_1+\ldots+x_nt_n\right)\right]^p,$$

where m is positive integer, h, v, p are any real or complex numbers independent of variables $x_1,\ldots,x_n$, and $|t_i|<1$, $i=1,\ldots,n$. In this section, we further give generalization of (5.2.1) by introducing another multivariable analogue

$$H^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p)}_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$$

of Gould and Hopper's polynomials [5] defined by generating relation

(5.2.2)
$$\sum_{n_1,\ldots,n_r}^{\infty} H^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p)}_{n_1,\ldots,n_r}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$= \exp\left[h_1t_1^{m_1}+\ldots+h_rt_r^{m_r}\right]\cdot\left[1+v_1x_1t_1,\ldots,v_rx_rt_r\right]^p,$$

where all $|t_i|<1$ and h_i, k_i, v_i and p are any real or complex numbers independent of all variables x_i while all m_i are non- negative integers $i=1,\ldots,r$.

Particularly for $h_1=\ldots=h_r=h$; $m_1=\ldots=m_r=m$ and $v_1=\ldots=v_r=v$, (5.2.2) reduces to (5.2.1). Also

A paper from this chapter, entitled "Another multivariable analogue of Gould and Hopper's polynomials" has been accepted for publication in Pure Math. Manuscript (Calcutta), 1992.

$$(5.2.3)\qquad \lim_{p\to\infty} \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, 1,\dots,1;\, p)}\left(\frac{x_1}{p},\dots,\frac{x_r}{p}\right)$$

$$= g_{n_1}^{m_1}(x_1, h_1)\dots g_{n_r}^{m_r}(x_r, h_r),$$

and

$$(5.2.4)\qquad \lim_{p\to\infty} \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, \frac{1}{p},\dots,\frac{1}{p};\, p)}(x_1,\dots,x_r)$$

$$= g_{n_1}^{m_1}(x_1, h_1)\dots g_{n_r}^{m_r}(x_r, h_r),$$

where $g_n^{\gamma}(x, h)$ are generalized Hermite polynomials of Gould and Hopper's [5], defined by

$$(5.2.5)\qquad \sum_{n=0}^{\infty} g_n^{\gamma}(x, h)\frac{t^n}{n!} = e^{xt+ht^{\gamma}}$$

5.3 EXPLICIT FORM . Starting with the generating relation (5.2.2), we derive the following explicit form for the above polynomials

$$(5.3.1)\qquad \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

$$= \sum_{i_1=0}^{[n_1/m_1]}\dots\sum_{i_r=0}^{[n_r/m_r]} (-1)^{n_1+\dots+n_r}(-p)_{n_1+\dots+n_r}(v_1x_1)^{n_1}\dots(v_rx_r)^{n_r}$$

$$\frac{(-n_1)_{m_1i_1}\dots(-n_r)_{m_ri_r}}{(1+p-n_1-\dots-n_r)_{m_1i_1+\dots+m_ri_r}}\;\frac{\left[\dfrac{h_1}{(-v_1x_1)^{m_1}}\right]^{i_1}}{i_1!}\dots\frac{\left[\dfrac{h_r}{(-v_rx_r)^{m_r}}\right]^{i_r}}{i_r!},$$

which can be further written in the form of the generalized multiple hypergeometric function of Srivastava and Daoust [6] (Also see Srivastava and Manocha [7, p.64, eq. (18) to (20)])

$$(5.3.2)\qquad \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

$$= (-1)^{n_1+\dots+n_r}(v_1x_1)^{n_1}\dots(v_rx_r^{n_r}(-p)_{n_1+\dots+n_r}$$

$$F_{1:0;\dots;0}^{0:1;\dots;1}\left(\begin{matrix}-:[-n_1:m_1];\dots;[-n_r:m_r];\\ [1+p-n_1-\dots-n_r:m_1,\dots,m_r]:-;\dots;-;\end{matrix}\;\frac{h_1}{(-v_1x_1)^{m_1}},\dots,\frac{h_r}{(-v_rx_r)^{m_r}}\right).$$

5.4 RECURRENCE RELATION. Making an appeal to generating relation (5.2.2), we get the following recurrence relation :

$$(5.4.1)\qquad \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p+1)}(x_1,\dots,x_r)$$

$$= \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

$$+ n_1 v_1 x_1 \, \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1-1,\,n_2,\dots,n_r}(x_1,\dots,x_r)$$

$$+\dots+ n_r v_r x_r \, \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r-1}(x_1,\dots,x_r)$$

5.5 <u>DIFFERENTIALS AND THEIR APPLCATIONS</u>. Starting with the generating relation (5.2.2), we derive

(5.5.1)
$$\frac{\partial}{\partial x_1} \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= p\, n_1 v_1 \, \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-1)}_{n_1-1,\,n_2,\dots,n_r}(x_1,\dots,x_r)$$

which can be written in the following form :

(5.5.2)
$$\frac{\partial}{\partial x_i} \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= p\, n_i v_i \, \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-1)}_{n_1,\dots,n_{i-1},\,n_i-1,\,n_{i+1},\dots,n_r}(x_1,\dots,x_r)$$

$i = 1,\dots, r$.

By repeating applications of (5.5.2), we further derive

(5.5.3)
$$\frac{\partial^s}{\partial x_i^s} \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$\frac{\Gamma(p+1)}{\Gamma(p-s+1)} \frac{(n_i)!}{(n_i-s)!} v_i^s \, \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s)}_{n_1,\dots,n_{i-1},\,n_i-s,\,n_{i+1},\dots,n_r}(x_1,\dots,x_r)$$

$i = 1,\dots, r$ and s is non-negative integer.

Also

(5.5.4)
$$\frac{\partial^{s_1+\dots+s_r}}{\partial x_1^{s_1}\dots\partial x_r^{s_r}} \mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$\frac{\Gamma(p+1)\, n_1!\dots n_r!}{\Gamma(p-s_1-\dots-s_r+1)} \frac{v_1^{s_1}}{(n_1-s_1)!} \cdots \frac{v_r^{s_r}}{(n_r-s_r)!}$$

$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s_1-\dots-s_r)}_{n_1-s_1,\dots,n_r-s_r}(x_1,\dots,x_r)$$

where $s_1,\dots, s_r$ are non-negative integers.

In (5.5.3), replacing x_i by $1/x_i$ and making an appeal to well known result due to Chandel and Agarwal [1, p.88, (3.2)] (Also see earlier reference due to Edwards [4, p.506 Misc. Ex. No. 15])

(5.5.5)
$$e^{t\Omega}\{f(x)\} = f\left(\frac{x}{1-xt}\right), \quad \Omega = x^2 \frac{\partial}{\partial x}$$

and finally replacing again x_i by $1/x_i$, we derive

(5.5.6)
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_{i-1},x_i+y,x_{i+1},\dots,x_r)$$
$$=\sum_{s=0}^{\min[n_i,\,p]}\frac{(v_i\,y)^s\,\Gamma(p+1)}{\Gamma(p-s+1)}\binom{r}{s}$$
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s)}_{n_1,\dots,n_{i-1},\,n_i-s,\,n_{i+1},\dots,n_r}(x_1,\dots,x_r)\,,$$

$i=1,\dots,r.$

Taking $x_i = 0$ and replacing y by x_i, we finally obtain

(5.5.7)
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r)}(x_1,\dots,x_r)$$
$$=\sum_{s=0}^{\min[n_i,\,p]}\frac{(v_i\,x_i)^s}{\Gamma(p-s+1)}\binom{n_i}{s}$$
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s)}_{n_1,\dots,n_{i-1},\,n_i-s,\,n_{i+1},\dots,n_r}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_r)$$

$i=1,\dots,r.$

Applying the same techniques in (5.5.4), we derive

(5.5.8)
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1+y_1,\dots,x_r+y_r)$$
$$=\sum_{s_1=0}^{\min(n_1,\,p)}\cdots\sum_{s_r=0}^{\min(n_r,\,p)}\binom{n_1}{s_1}\cdots\binom{n_r}{s_r}\frac{\Gamma(p+1)\,(v_1\,x_1)^{s_1}\cdots(v_r\,x_r)^{s_r}}{\Gamma(p-s_1-\dots-s_r+1)}$$
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s_1-\dots-s_r)}_{n_1-s_1,\dots,n_r-s_r}(x_1,\dots,x_r)$$

Taking $x_1 = \dots = x_r = 0$ replacing $y_1,\dots,y_r$ by $x_1,\dots,x_r$ respectively, we finally arrive at

(5.5.9)
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$
$$=\sum_{s_1=0}^{\min(n_1,\,p)}\cdots\sum_{s_r=0}^{\min(n_r,\,p)}\binom{n_1}{s_1}\cdots\binom{n_r}{s_r}\frac{\Gamma(p+1)\,(v_1\,x_1)^{s_1}\cdots(v_r\,x_r)^{s_r}}{\Gamma(p-s_1-\dots-s_r+1)}$$
$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p-s_1-\dots-s_r)}_{n_1-s_1,\dots,n_r-s_r}(0,\dots,0)\,.$$

5.6 <u>**OTHER RESULTS.**</u> Rewriting the generating relation (5.2.2) in the form

$$\left[1+v_1\,x_1\,t_1+\dots+v_n\,x_n\,t_n\right]^p = \exp\left[-\left\{h_1\,t_1^{m_1}+\dots+h_r\,t_r^{m_r}\right\}\right]$$

$$\sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

and comparing the coefficients of $t_1^{n_1} \dots t_r^{n_r}$ both the sides, we derive

(5.6.1)
$$x_1^{n_1} \dots x_r^{n_r} = \frac{n_1! \dots n_r!}{(-v_1)^{n_1} \dots (-v_r)^{n_r} (-p)_{n_1+\dots+n_r}}$$

$$\sum_{s_1=0}^{[n_1/m_1]} \cdots \sum_{s_r=0}^{[n_r/m_r]} \frac{(-1)^{s_1+\dots+s_r}}{(n_1-s_1m_1)!\dots(n_r-s_rm_r)!}\,\frac{h_1^{s_1}}{s_1!}\cdots\frac{h_r^{s_r}}{s_r!}$$

$$\mathbf{H}_{n_1-s_1m_1,\dots,n_r-s_rm_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r).$$

Making an appeal to generating relation (5.2.2), we also derive

(5.6.2)
$$\mathbf{H}_{n_1,\dots,n_r}^{(h_1+h'_1,\dots,h_r h'_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p+p')}(x_1,\dots,x_r)$$

$$= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1}\cdots\binom{n_r}{k_r} \mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

$$\mathbf{H}_{k_1,\dots,k_r}^{(h'_1,\dots,h'_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p')}(x_1,\dots,x_r)$$

5.7 GENERALIZATION RELATION. Consider

(5.7.1)
$$\sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{R}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}(x_1,\dots,x_r)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$= \exp\left[h_1 t_1^{m_1} + \dots + h_r t_r^{m_r}\right] G(v_1 x_1 t_1 + \dots + v_r x_r t_r)$$

where

(5.7.2)
$$\mathbf{G}(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n \ , \quad \gamma_n \neq 0,$$

and $m_1,\dots,m_r$ are non-negative integers, $h_1,\dots,h_r$; $v_1,\dots,v_r$ are any real or complex numbers independent of variable $x_1,\dots,x_r$.

For $\gamma_n = (-1)^n (-p)_n$, (5.7.1) reduces to (5.2.2); while for $\gamma_n = 1$, (5.7.1) defines new polynomials of several variables by generating relation

(5.7.3)
$$\exp\left[h_1 t_1^{m_1} + \dots + h_r t_r^{m_r} + v_1 x_1 t_1 + \dots + v_r x_r t_r\right]$$

$$= \sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{E}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}(x_1,\dots,x_r)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

which for $v_1 = \dots = v_r = 1$, further gives

(5.7.4)
$$\mathbf{E}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,1,\dots,1)}(x_1,\dots,x_r)$$
$$= g_{n_1}^{m_1}(x_1, h_1)\dots g_{n_r}^{m_r}(x_r, h_r),$$

where $g_n^m(x, h)$ are Gould and Hopper's polynomials [5] (see also Srivastava and Manocha [7, p.86, eqn. (27)]).

Starting with the generating relation (5.7.1), we derive the following explicit form for the generalized multivariable polynomials :

(5.7.5)
$$\mathbf{R}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}(x_1,\dots,x_r)$$
$$= \frac{(v_1 x_1)^{n_1}}{n_1!}\dots\frac{(v_r x_r)^{n_r}}{n_r!}\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\gamma_{n_1+\dots+n_r-m_1k_1-\dots-m_rk_r}$$
$$\times\frac{(-n_1)_{m_1k_1}}{k_1!}\dots\frac{(-n_r)_{m_rk_r}}{k_r!}\left[\frac{h_1}{(-v_1x_1)^{m_1}}\right]^{k_1}\dots\left[\frac{h_r}{(-v_rx_r)^{m_r}}\right]^{k_r}.$$

Differentiating (5.7.1) partially w.r.t. x_i and t_i separately and eliminating G', we obtain

(5.7.6)
$$x_i\frac{\partial}{\partial x_i}\mathbf{R}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}(x_1,\dots,x_r)$$
$$= n_i\,\mathbf{R}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}(x_1,\dots,x_r)$$
$$- h_i m_i n_i (n_i - 1)\dots(n_i - m_i + 1)$$
$$\mathbf{R}_{n_1,\dots,n_{i-1},\,n_i-m_i,\,n_{i+1},\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\;v_1,\dots,v_r)}(x_1,\dots,x_r),$$

$i = 1,\dots, r$.

An appeal to the above result directly gives the following result for the polynomials defined by (5.2.2) and (5.7.3) respectively

(5.7.7)
$$x_i\frac{\partial}{\partial x_i}\mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}(x_1,\dots,x_r)$$
$$= n_i\,\mathbf{H}_{n_1,\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}(x_1,\dots,x_r)$$
$$- h_i m_i n_i (n_i - 1)\dots(n_i - m_i + 1)$$
$$\mathbf{H}_{n_1,\dots,n_{i-1}\,.\,n_i-m_1,\,n_{i+1},\dots,n_r}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r\,;\,p)}(x_1,\dots,x_r),$$

$i = 1,\dots, r$.
and

$$(5.7.8)\qquad x_i \frac{\partial}{\partial x_i} \mathbf{E}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r),$$

$$-h_i\, m_i\, n_i\,(n_i-1)\dots(n_i-m_i+1)$$

$$\mathbf{E}_{n_1,\, n_{i-1},\, n_i - m_i,\, n_{i+1}\dots,\, n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

$i = 1,\dots, r.$

5.8 ANOTHER MULTIVARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS.

Recently Chandel and Sahgal [3] introduced a multivariable analogue of Gould and Hopper's polynomials [5] through their generating relation

$$(5.8.1)\qquad \sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{P}_{n_1,\dots,n_r}^{(m_1,\dots,m_r;\, M_1,\dots,M_r;\, h_1,\dots,h_r;\, p)}(x_1,\dots,x_r)\, \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$= \left[1 + m_1 x_1 t_1 + h_1 t_1^{M_1} + \dots + m_r x_r t_r + h_r t_r^{M_r}\right]^p,$$

where $M_1,\dots, M_r$ are positive integers and $m_1,\dots, m_r$, $h_1,\dots, h_r$ are any real or complex numbers independent of variables $x_1,\dots, x_r$. They also gave the generalization of (5.8.1) in the following form :

$$(5.8.2)\qquad \sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{g}_{n_1,\dots,n_r}^{(m_1,\dots,m_r;\, M_1,\dots,M_r;\, h_1,\dots,h_r)}(x_1,\dots,x_r)\, \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$= G\left(m_1 x_1 t_1 + h_1 t_1^{M_1} + \dots + m_r x_r t_r + h_r t_r^{M_r}\right)$$

where G(z) is given by (5.7.2).

Motivated by the above works, in this part of the present chapter we also introduced another multivariable analogue

$$\mathbf{S}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)$$

of Gould and Hopper's polynomials [5], defined by generating relation

$$(5.8.3)\qquad \sum_{n_1,\dots,n_r=0}^{\infty} \mathbf{S}_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p)}(x_1,\dots,x_r)\, \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$= \left[1 + v_1 x_1 t_1 + \dots + v_r x_r t_r\right]^p G(h_1 t_1^{m_1} + \dots + h_r t_r^{m_r}$$

where $|t_i| < 1$ and all parameters h_i, v_i, p are unrestricted in general but independent of all variables x_i, while m_i are non-negative integers, i=1,..., r, and G(z) is given by (5.7.2).

For $\gamma_n = \frac{1}{n!}$, (5.8.3) reduces to (5.2.2) while for $\gamma_n = \frac{(-1)^n (q)_n}{n!}$, (5.8.3) defines a new set of polynomials by

(5.8.4) $$\left[1+v_1x_1t_1+\ldots+v_rx_rt_r\right]^p\left[1+h_1t_1^{m_1}+\ldots+h_rt_r^{m_r}\right]^{-q}$$

$$=\sum_{n_1,\ldots,n_r=0}^{\infty}\mathbf{B}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p,q)}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

Replacing h_i by $\frac{-h_i}{q}$ and taking lim $q \to \infty$ (5.8.4) reduces to (5.2.2). Hence

(5.8.5) $$\lim_{q\to\infty}\mathbf{B}_{n_1,\ldots,n_r}\left(\frac{-h_1}{q},\ldots,\frac{-h_r}{q};\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p,q\right)(x_1,\ldots,x_r)$$

$$=\mathbf{H}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p)}(x_1,\ldots,x_r).$$

Another special case of (5.8.3) can be obtained by replacing x_i by $\frac{x_i}{p}$, $i = 1,\ldots, r$ and taking lim$p \to \infty$, in the following form :

(5.8.6) $$\sum_{n_1,\ldots,n_r=0}^{\infty}\mathbf{A}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r)}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

$$=\exp\left[v_1x_1t_1+\ldots+v_rx_rt_r\right].\,G(h_1t_1^{m_1}+\ldots+h_rt_r^{m_r}).$$

5.9 <u>EXPLICIT FORM</u>. Making an appeal to generating relation (5.8.3) we derive the following explicit form

(5.9.1) $$\mathbf{S}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p)}(x_1,\ldots,x_r)$$

$$=(-v_1x_1)^{n_1}\ldots(-v_rx_r)^{n_r}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{\gamma_{k_1+\ldots+k_r}(-n_1)_{m_1k_1}\ldots(-n_r)_{m_rk_r}}{(1+p-(n_1+\ldots+n_r))_{n_1k_1+\ldots+n_rk_r}}$$

$$.\,(-p)_{n_1+\ldots+n_r}(1)_{k_1+\ldots+k_r}\frac{(-v_1x_1)^{m_1k_1}}{k_1!}\cdots\frac{(-v_rx_r)^{m_rk_r}}{k_r!},$$

where γ_k is defined by (5.7.2).

5.10 <u>APPLICATION OF GENERATING RELATION</u> . Making an appeal to generating relation (5.8.3) we derive

(5.10.1) $$\mathbf{S}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,p+q)}(x_1,\ldots,x_r)$$

$$=\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}(-p)_{k_1+\ldots+k_r}(-v_1x_1)^{k_1}\ldots(-v_rx_r)^{k_r}$$

$$\mathbf{S}_{n_1-k_1,\ldots,n_r-k_r}^{(h_1,\ldots,h_r;\,m_1,\ldots,m_r;\,v_1,\ldots,v_r;\,q)}(x_1,\ldots,x_r).$$

An appeal to (5.8.3) also shows that

$$(5.10.2)\qquad \frac{\partial}{\partial x_i} S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1,\dots,x_r)$$

$$= p\, v_i\, n_i\, S_{n_1,\dots,n_{i-1},\,n_i-1,\,n_{i+1},\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p-1)}(x_1,\dots,x_r),\ i=1,\dots,r.$$

Repeated applications of (5.10.2) further give

$$(5.10.3)\qquad \frac{\partial^s}{\partial x_i^s} S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1,\dots,x_r)$$

$$= \frac{\Gamma(p+1)\,\Gamma(n_i+1)}{\Gamma(p-s+1)\,\Gamma(n_i-s+1)}\, v_i^s$$

$$S_{n_1,\dots,n_{i-1},\,n_i-s,\,n_{i+1},\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p-s)}(x_1,\dots,x_r),\ i=1,\dots,r.$$

Replacing x_i by $\frac{1}{x_i}$ in (5.10.3) and making an appeal to well known result (5.5.5), we derive

$$(5.10.4)\qquad S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1,\dots,x_r)$$

$$= \sum_{r=0}^{\min(p,\,n_i)} \binom{n_i}{s} (-p)_s\, (v_i\ t)^s\, S_{n_1,\dots,n_{i-1},\,n_i-s,\,n_{i+},\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p-s)}(x_1,\dots,x_r)$$

$i=1,\dots,r$

An appeal to (5.10.3) also shows that

$$(5.10.5)\qquad \frac{\partial^{s_1+\dots+s_r}}{\partial x_1^{s_1}\dots\partial x_r^{s_r}} S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1,\dots,x_r)$$

$$= \frac{\Gamma(p+1)\, n_1!\dots n_r!}{(n_1-s_1)!\dots(n_r-s_r)!}\ \frac{v_1^{s_1}\dots v_r^{s_r}}{\Gamma(p-s_1-\dots-s_r+1)}$$

$$S_{n_1-s_1,\dots,n_r-s_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p-s_1,\dots,p-s_r)}(x_1,\dots,x_r).$$

Replacing each x_i by $\frac{1}{x_i}$, $i=1,\dots,r$ in (5.10.5) and making an appeal to (5.5.5) we derive

$$(5.10.6)\qquad S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1+y_1,\dots,x_r+y_r)$$

$$= \sum_{s_1=0}^{\min(n_1,\,p)} \dots \sum_{s_r=0}^{\min(n_r,\,p)} (v_1 y_1)^{s_1} \dots (v_r y_r)^{s_r} \binom{n_1}{s_1}\dots\binom{n_r}{s_r}$$

$$\frac{\Gamma(p+1)}{\Gamma(p-s_1-\dots-s_r+1)}\, S_{n_1-s_1,\dots,n_r-s_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p-s_1-,\dots,-s_r)}(x_1,\dots,x_r)$$

Choosing $x_1=\dots=x_r=0$ and replacing each y_i by x_i, $i=1,\dots,r$, we derive

$$(5.10.7)\qquad S_{n_1,\dots,n_r}^{(h_1,\dots,h_r;\,m_1,\dots,m_r;\,v_1,\dots,v_r;\,p)}(x_1,\dots,x_r)$$

$$= \sum_{s_1=0}^{\min(n_1,p)} \cdots \sum_{s_r=0}^{\min(n_r,p)} (v_1 x_1)^{s_1} \ldots (v_r x_r)^{s_r} \binom{n_1}{s_1} \cdots \binom{n_r}{s_r}$$

$$\frac{\Gamma(p+1)}{\Gamma(p-s_1-\ldots-s_r+1)} \mathbf{S}_{n_1-s_1,\ldots,n_r-s_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p-s_1-,\ldots,-s_r)} (0,\ldots,0)$$

5.11 RECURRENCE RELATIONS. Making an appeal to generating relation (5.8.3) we derive the following recurrence relations:

(5.11.1)
$$\mathbf{S}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p+1)} (x_1,\ldots,x_r)$$

$$= \mathbf{S}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p)} (x_1,\ldots,x_r)$$

$$+ v_1 x_1 n_1 \mathbf{S}_{n_1-1,\, n_2,\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p)} (x_1,\ldots,x_r)$$

$$+ \ldots + v_r x_r n_r \mathbf{S}_{n_1,\ldots,n_{r-1},\, n_r-1}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p)} (x_1,\ldots,x_r),$$

and

(5.11.2)

$$\frac{h_j m_j n_j!}{(n_j-m_j+1)!} \mathbf{S}_{n_1,\ldots,n_{i-1},\, n_i+1,\, n_{i+1},\ldots,n_{j-1},\, n_j-m_j+1,\, n_{j+1},\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p)} (x_1,\ldots,x_r).$$

$$- \frac{h_i m_i n_i!}{(n_i-m_i+1)!} \mathbf{S}_{n_1,\ldots,n_{i-1},\, n_i-m_i+1,\, n_{i+1},\ldots,n_{j-1},\, n_j+1,\, n_{j+1},\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p)} (x_1,\ldots,x_r)$$

$$= \frac{p v_i x_i h_j m_j n_j!}{(n_j-m_j+1)!} \mathbf{S}_{n_1,\ldots,n_{j-1},\, n_j-m_j+1,\, n_{j+1},\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p-1)} (x_1,\ldots,x_r) -$$

$$- \frac{p v_j x_j h_i m_i n_i!}{(n_i-m+1)!} \mathbf{S}_{n_1,\ldots,n_{i-1},\, n_i-m_i+1,\, n_{i+1},\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p-1)} (x_1,\ldots,x_r)$$

where $i, j \in \{1,\ldots,r\}$ and $i \neq j$.

5.12 SPECIAL CASE (5.8.4). An appeal to (5.8.4) shows that

(5.12.1)
$$\mathbf{B}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p+p',\, q+q')} (x_1,\ldots,x_r)$$

$$= \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r}$$

$$\mathbf{B}_{n_1-k_1,\ldots,n_r-k_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p,\, q)} (x_1,\ldots,x_r)$$

$$\mathbf{B}_{k_1,\ldots,k_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p',\, q')} (x_1,\ldots,x_r),$$

(5.12.2)
$$\mathbf{B}_{n_1,\ldots,n_r}^{(h_1,\ldots,h_r;\, m_1,\ldots,m_r;\, v_1,\ldots,v_r;\, p+1,\, q)} (x_1,\ldots,x_r)$$

$$= \mathbf{B}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p,q)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$+ \sum_{i=1}^{r} v_i x_i n_i \, \mathbf{B}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p,q)}_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r}(x_1,\dots,x_r)$$

and

(5.12.3) $$\mathbf{B}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p,q-1)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= \mathbf{B}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p,q)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$+ \sum_{i=1}^{r} h_i n_i (n_i-1)\dots(n_i-m_i+1)$$

$$\mathbf{B}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r;\, p,q)}_{n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_r}(x_1,\dots,x_r).$$

5.13 <u>SPECIAL CASE (5.8.6)</u>. An appeal to (5.8.6) gives

(5.13.1) $$\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1+y_1,\dots,x_r+y_r)$$

$$= \sum_{k_1=0}^{n_1} \dots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1}\dots\binom{n_r}{k_r}(v_1x_1)^{k_1}\dots(v_rx_r)^{k_r}$$

$$\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1-k_1,\dots,n_r-k_r}(y_1,\dots,y_r)$$

(5.13.2) $$\frac{\partial^s}{\partial x_i^s}\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= (v_i)^s n_i(n_i-1)\dots(n_i-s+1)\,\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},n_i-s,n_{i+1},\dots,n_r}(x_1,\dots,x_r)$$

$i = 1,\dots, r$.

Now replacing x_i by $\frac{1}{x_i}$, $i = 1,\dots, r$ in (5.13.2) and making an appeal to (5.5.5) we derive

(5.13.3) $$\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_r)$$

$$= \sum_{s=0}^{n_i} (v_i)^s \binom{n_i}{s} t^s$$

$$\mathbf{H}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},n_i-r,n_{i+1},\dots,n_r}(x_1,\dots,x_r).$$

Taking $x_i = 0$ and replacing t by x_i, we further derive

(5.13.4) $$\mathbf{A}^{(h_1,\dots,h_r;\, m_1,\dots,m_r;\, v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= \sum_{s=0}^{n_i} (v_i)^s \binom{n_i}{s} t^s$$

$$\mathbf{H}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},\,n_i-r,\,n_{i+1},\dots,n_r}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_r)$$

Again starting from (5.8.6) we derive

(5.13.5)
$$\frac{\partial^{s_1+\dots+s_r}}{\partial x_1^{s_1}\dots\partial x_r^{s_r}}\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= \prod_{i=1}^{r} (v_i)^{s_i}\, n_i(n_i-1)\dots(n_i-s_i+1)\ \mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1-s_1,\dots,n_r-s_r}(x_1,\dots,x_r)$$

Replacing each x_i by $\frac{1}{x_i}$, $i = 1,\dots, r$ and making an appeal to (5.5.5), we derive

(5.13.6)
$$\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1+y_1,\dots,x_r+y_r)$$

$$= \sum_{s_1=0}^{n_1}\dots\sum_{s_r=0}^{n_r}\prod_{i=1}^{r}(v_i\,y_i)^{s_i}\binom{n_i}{s_i}$$

$$\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1-s_1,\dots,n_r-s_r}(x_1,\dots,x_r)$$

Choosing $x_i = \dots = x_r = 0$ and replacing each y_i by x_i, $i = 1,\dots, r$, we further derive

(5.13.7)
$$\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_r}(x_1,\dots,x_r)$$

$$= \sum_{s_1=0}^{n_1}\dots\sum_{s_r}^{n_r}\prod_{i=1}^{r}(v_i\,x_i)^{s_i}\binom{n_i}{s_i}$$

$$\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_r}(0,\dots,0)$$

An appeal to (5.8.6) also gives

(5.13.8)
$$\frac{h_j\,m_j\,n_j\,!}{(n_j-m_j+1)\,!}\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},\,n_{i+1},\dots,n_{j-1},\,n_j-m_j+1,\dots,n_r}(x_1,\dots,x_r)$$

$$-\frac{h_i\,m_i\,n_i\,!}{(n_i-m_i+1)\,!}\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},\,n_i-m_i+1,\,n_{i+1},\dots,n_{j-1},\,n_j+1,\,n_{j+1},\dots,n_r}(x_1,\dots,x_r)$$

$$= \frac{v_i\,x_i\,h_j\,n_j\,m_j\,!}{(n_j-m_j+1)\,!}\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_{j-1},\,n_j-m_j+1,\,n_{j+1},\dots,n_r}(x_1,\dots,x_r)$$

$$-\frac{v_j\,x_j\,h_i\,m_i\,n_i\,!}{(n_i-m_i+1)\,!}\mathbf{A}^{(h_1,\dots,h_r\,;\,m_1,\dots,m_r\,;\,v_1,\dots,v_r)}_{n_1,\dots,n_{i-1},\,n_i-m_i+1,\,n_{i+1},\dots,n_r}(x_1,\dots,x_r)$$

where $i, j \in \{1,\dots, r\}$ and $i \neq j$.

5.14 A TWO VARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS AS A SPECIAL CASE OF (5.2.2).

For $r = 2, x_1 = x, x_2 = y, h_1 = h, h_2 = H, t_1 = t, t_2 = T, v_1 = v, v_2 = V, m_1 = m, m_2 = M$, from (5.2.2), we define two variable analogue of Gould and Hopper's polynomials by generating relation.

(5.14.1) $$\exp\left[h\, t^m + H\, T^M\right] . \left[1 + v\,x\,t + V\,y\,T\right]^P = \sum_{n,k=0}^{\infty} \mathbf{H}_{n,k}^{(h, H; m, M; v, V; P)}(x, y),$$

where m, M are positive integers and h, H, v, V, p are any real or complex numbers independent of variables x and y.

For H = h, M = m, V = v, (5.14.1) further reduces to the polynomials of Chandel, Agrawal and Kumar [2(a), p.63, (1.3)] defined by

(5.14.2) $$\exp\left[h\,(t^m + T^m)\right] . \left(1 + v\,x\,t + V\,y\,T\right)^p = \sum_{n,k=0}^{\infty} \mathbf{H}_{n,k}^{(h, m, v, p)}(x, y)\,\frac{t^n}{n!}\,\frac{T^k}{k!}$$

where m is positive integer, h, v, p are any real or complex numbers independent of vartiables x and y.

From (5.14.1), it is clear that

(5.14.3) $$\lim_{p\to\infty} \mathbf{H}_{n,k}^{(h, H; m, M; 1/p, 1/p : p)}(x, y) = g_n^m(x, h)\, g_k^M(y, H)$$

also

(5.14.4) $$\lim_{p\to\infty} \mathbf{H}_{n,k}^{(h, H; m, M; 1, 1; p)}(x/p, y/p) = g_n^m(x, h) . g_k^M(y, H),$$

where $g_n^r(x, h)$ are Gould and Hopper's polynomials [4].

5.15. EXPLICIT FORM. Starting with the generating relation (5.14.1), we obtain the following explicit form

(5.15.1) $$\mathbf{H}_{n,k}^{(h, H; m, M; v, V; p)}(x, y) = (-p)_{n+k}\, x^n\, y^k\, (-v)^n\, (-V)^k$$
$$\sum_{i=0}^{[n/m]} \sum_{j=0}^{[k/m]} \frac{h^i}{i!}\,\frac{H^j}{j!}\,\frac{(-n)_{m_i}\,(-k)_{m_j}}{(p+1-n-k)_{m_i+m_j}} \left[\frac{-1}{xV}\right]^{m_i} . \left[-\frac{1}{y\,V}\right]^{M_j},$$

which can be further written as in the following form of generalized Kampé de Fériet function of Srivastava ans Daoust [6],

(5.15.2) $$\mathbf{H}_{n,k}^{(h, H; m, M; v, V; p)}(x, y) = (-p)_{n+k}\, x^n\, y^k\, (-v)^n\, (-V)^k$$

$$\mathbf{F}\,{}^{0:1;1}_{1:0;0}\left(\begin{matrix} -:[-n:m];[-k:M] \\ [p+1-n-k:m,M]:-:-; \end{matrix}\;\left(\frac{-1}{xv}\right)^{m}\;\left(\frac{-1}{yV}\right)^{n}\right).$$

5.16 Differentials and their applications : Starting from generating relation (5.14.1), we derive

(5.16.1)
$$\frac{\partial}{\partial x}\,\mathbf{H}^{(h,H:m,M;v,V:p)}_{n,k}(x,y)$$

$$= n\,v\,p\,\mathbf{H}^{(h,H;m,M;v,V;p-1)}_{n-1,k}(x,y)$$

which on repeated applications, further gives.

(5.16.2)
$$\frac{\partial^r}{\partial x^r}\,\mathbf{H}^{(h,H;m,M;v,V;p)}_{n,k}(x\ y)$$

$$= \frac{v^r\,n!}{(n-r)!}\,\frac{\Gamma(p+1)}{\Gamma(p-r+1)}\,\mathbf{H}^{(h,H;m,M;v,V;p-r)}_{n-r,k}(x,y).$$

From (5.14.1), we also obtain

(5.16.3)
$$\frac{\partial}{\partial y}\,\mathbf{H}^{(h,H;m,M;v,V;p)}_{n,k}(x,y)$$

$$= k\,V\,p\,\mathbf{H}^{(h,H;m,M;v,V;p-1)}_{n,k-1}(x,y)$$

which on induction gives

(5.16.4)
$$\frac{\partial^r}{\partial y^r}\,\mathbf{H}^{(h,H;m,M;v,V;p)}_{n,k}(x,y)$$

$$= \frac{V^r\,k!\,\Gamma(p+1)}{(k+r)!\,\Gamma(p+1-r)}\,\mathbf{H}^{(h,H;m,M;v,V;p-r)}_{n,k-r}(x,y).$$

Replacing x by 1/x in (5.16.1) and applying well known result due to Chandel and Agrawal [1, p.88 (3.2)] (Also see earlier reference due to Edwards [4, p.506 Misc. No. 15])

$$e^{t\,\Omega_x}\{f(x)\} = f\left\{\frac{x}{1-xt}\right\},\qquad \Omega_x = x^2\frac{\partial}{\partial x},$$

and finally replacing x by 1/x, t by z, we get

(5.16.5)
$$\mathbf{H}^{(h,H;m,M;v,V;p)}_{n,k}(x+z,y)$$

$$= \sum_{r=0}^{\min(n,p)} v^r z^r \binom{n}{r}\frac{\Gamma(p+1)}{\Gamma(p+1-r)}\,\mathbf{H}^{(h,H;m,M;v,V;p-r)}_{n-r,k}(x,y).$$

Taking x = 0 and replacing z by x, we further derive

(5.16.6)
$$\mathbf{H}^{(h,H;m,M;v,V;p)}_{n,k}(x,y)$$

$$= \sum_{r=0}^{\min(n,p)} v^r x^r \binom{n}{r}\frac{\Gamma(p+1)}{\Gamma(p+1-r)}\,\mathbf{H}^{(h,H;m,M;v,V;p-r)}_{n-r,k}(0,y).$$

Similarly replacing y by 1/y in (5.16.1) and applying the same techniques, we derive

(5.16.7) $$\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y+z) = \sum_{r=0}^{\min(p,k)} \frac{V^r \Gamma(p+1)}{\Gamma(p+1-r)} z^r \binom{k}{r} \mathbf{H}_{n,k-r}^{(h,H;m,M;v,V;p-r)}(x,y).$$

Taking y = o and replacing z by y, we derive

(5.16.8) $$\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) = \sum_{r=0}^{\min(p,k)} \frac{V^r \Gamma(p+1)}{\Gamma(p+1-r)} \binom{k}{r} y^r \mathbf{H}_{n,k-r}^{(h,H;m,M;v,V;p-r)}(x,0).$$

From generating relation (5.14.1), we obtain

(5.16.9) $$\frac{\partial^2}{\partial y\,\partial x} \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) = n\,k\,v\,V\,p\,(p-1)\, \mathbf{H}_{n-1,k-1}^{(h,H;m,M;v,V;p-2)}(x,y)$$

which on repeated applications gives

(5.16.10) $$\frac{\partial^{r+s}}{\partial y^s\,\partial x^r} \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) = \frac{n!\,k!\,v^r V^s}{(n-r)!\,(k-s)!} \frac{\Gamma(p+1)}{\Gamma(p+1-r-s)} \mathbf{H}_{n-r,k-s}^{(h,H;m,M;v,V;p-r-s)}(x,y).$$

Replacing x by 1/x, y by 1/y in (5.16.10) and applying the techniques of (5.16.5), we establish

(5.16.11) $$\sum_{r=0}^{\min(n,p)} \sum_{s=0}^{\min(k,p)} \binom{n}{r}\binom{k}{s} \frac{\Gamma(p+1)}{\Gamma(p+1-r-s)} v^r V^s t^r T^s \mathbf{H}_{n-r,k-s}^{(h,H;m,M;v,V;p-r-s)}(x,y) = \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x+t,y+T)$$

Taking x = y = 0 and replacing t by x, T by y, we have

(5.16.12) $$\sum_{r=0}^{\min(n,p)} \sum_{s=0}^{\min(k,p)} \binom{n}{r}\binom{k}{s} \frac{\Gamma(p+1)}{\Gamma(p+1-r-s)} v^r V^s x^r y^s \mathbf{H}_{n-r,k-s}^{(h,H;m,M;v,V;p-r-s)}(0,0) = \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$$

5.17 <u>Other Results :</u> An appeal to generating relation (5.14.1), gives

(5.17.1) $$\mathbf{H}_{n,k}^{(h+h',H+H';m,M;v,V;p+p')}(x,y) = \sum_{n'=0}^{n} \sum_{k'=0}^{k} \binom{n}{n'}\binom{k}{k'} \mathbf{H}_{n-n',k-k'}^{(h,H;m,M;v,V;p)}(x,y)\, \mathbf{H}_{n',k'}^{(h',H';m,M;v,V;p')}(x,y)$$

(5.17.2) $$\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) = \mathbf{H}_{n,k}^{(h,H;m,M;-v,-V;p)}(-x,-y).$$

Again from generating relation (5.14.1), we derive recurrence relation.

(5.17.3)
$$\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p+1)}(x,y)$$
$$= \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) + n v x\, \mathbf{H}_{n-1,k}^{(h,H;m,M;v,V;p)}(x,y) + k V y$$
$$\mathbf{H}_{n,k-1}^{(h,H;m,M;v,V;p)}(x,y)$$

From generating relation(5.14.1), we also derive

(5.17.4)
$$x^n y^k = \frac{k!\,n!}{(-v)^n (-V)^k (-p)_{n+k}}$$
$$\sum_{r=0}^{[n/m]} \sum_{s=0}^{[k/M]} \frac{(-h)^r (-H)^s}{r!\,s!\,(n-m_r)!\,(k-M_s)!} \mathbf{H}_{n-m_r}^{(h,H;m,M;v,V;p)}(x,y) \quad .$$

and

(5.17.5)
$$\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$$
$$= \sum_{r=0}^{n} \sum_{s=0}^{k} \binom{n}{r}\binom{k}{s} (-v x)^r (-V y)^s (-p+q)_{r+s}$$
$$\mathbf{H}_{n-r,k-s}^{(h,H;m,M;v,V;q)}(x,y)$$

5.18 <u>**Generalization.**</u> Consider

(5.18.1)
$$\exp\left(h t^m + H T^M\right) G(v x t + V y T)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y) \frac{t^n}{n!} \frac{T^k}{k!} \quad |t| < 1, |T| < 1.$$

where m, M are positive integers, v, V, h, H are arbitrary real or complex numbers independent of variables x and y ; and

(5.18.2)
$$G(z) = \sum_{n=0}^{\infty} \gamma_n z^n, \quad |z|<1.$$

Generating relation (5.18.1) may very well be regarded as the generalization of (5.14.1). Particularly,

for $\gamma_n = \dfrac{(-1)^n (-p)_n}{n!}$, (5.18.1) reduces to (5.14.1). For $\gamma_n = \dfrac{1}{n!}$, (5.18.1) gives

(5.18.3)
$$\exp\left[h t^m + H T^M\right] . \exp(v x t + V y T)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{g}_{n,k}^{(h,H;m,M;v,V)}(x,y) \frac{t^n}{n!} \frac{T^k}{k!},$$

which shows that

(5.18.4) $$\mathbf{g}_{n,k}^{(h,H;m,M;1,1)}(x,h) = g_n^m(x,h)\, g_k^m(y,H).$$

From (5.18.1), we derive

(5.18.5) $$\mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y) = n!\,k!\,(v\,x)^n (V\,Y)^k \,.$$

$$\sum_{r=0}^{[n/m]} \sum_{k=0}^{[k/M]} \gamma_{n+k-rm-mp} \binom{n+k-m\,r-m\,p}{k-mp} \frac{[h/(v\,x)^m]^r}{r!} \frac{[H/(V\,y)^m]^p}{p!} \,.$$

From (5.18.1), we also derive

(5.18.6) $$x \frac{\partial}{\partial x} \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y)$$

$$= n\, \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y) - m\,h\,n\,(n-1)\ldots(n-m+1)\, \mathbf{G}_{n-m,k}^{(h,H;m,M;v,V)}(x,y)$$

(5.18.7) $$y \frac{\partial}{\partial y} \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y) = k\, \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y)$$

$$- M\,H\,k\,(k-1)\ldots(k-M+1)\, \mathbf{G}_{n,k-M}^{(h,H;m,M;v,V)}(x,y)$$

From (5.18.6) and (5.18.7), we further derive

(5.18.8) $$\left(k\,x \frac{\partial}{\partial x} - n\,y \frac{\partial}{\partial y}\right) \mathbf{G}_{n,k}^{(h,H;m,M;v,V)}(x,y)$$

$$= n\,k \Big[M\,H\,(k-1)\ldots(k-M+1)\, \mathbf{G}_{n,k-M}^{(h,H;m,M;v,V)}(x,y)$$

$$- m\,h\,(n-1)\ldots(n-m+1)\, \mathbf{G}_{n-m,k}^{(h,H;m,M;v,V)}(x,y).$$

An appeal to (5.18.6), (5.18.7) and (5.18.8) gives the following results respectively for $\mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$

(5.18.9) $$x \frac{\partial}{\partial x} \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$$

$$= n\, \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) - m\,h\,n\,(n-1)\ldots(n-m+1)\, \mathbf{H}_{n-m,k}^{(h,H;m,M;v,V;p)}(x,y),$$

(5.18.10) $$y \frac{\partial}{\partial y} \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$$

$$= k\, \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y) - M\,H\,k\,(k-1)\ldots(k-M+1)\, \mathbf{H}_{n,k-M}^{(h,H;m,M;v,V;p)}(x,y)$$

and

(5.18.11) $$\left(k\,x \frac{\partial}{\partial x} - n\,y \frac{\partial}{\partial y}\right) \mathbf{H}_{n,k}^{(h,H;m,M;v,V;p)}(x,y)$$

$$= n\,k\left[M\,H\,(k-1)\ldots(k-M+1)\,\mathbf{H}_{n,\,k-M}^{(h,\,H\,;\,m,\,M\,;\,v,\,V\,;\,p)}(x\,,y\,)\right.$$

$$\left.-m\,h\,(n-1)\ldots(n-m+1)\,\mathbf{H}_{n-m,\,k}^{(h,\,H\,;\,m,\,M\,;\,v,\,V\,;\,p)}(x,y\,)\right].$$

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CHAPTER-VI

GENERATING RELATIONS INVOVLING HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

6.1 INTRODUCTION.

Chandel [1] established generating relations for Exton's multiple hypergeometric function ${}^{(k)}_{(1)}E^{(n)}_{D}$ [4] related to Lauricella's $F^{(n)}_{D}$, and for his own multiple hypergeometric function ${}^{(k)}_{(1)}E^{(n)}_{C}$ [1] related to Lauricella's $F^{(n)}_{C}$. Also Chandel and Gupta [2] introduced three intermediate Lauricella's function ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ and obtained generating relations involving them. Recently Chandel and Vishwakarma [3] introduced confluent hypergeometric functions of fourth possible intermediate Lauricella's hypergeometric function ${}^{(K)}F^{(n)}_{CD}$ of Karlsson [8] and obtained their generating relations.

In this chapter, for special interest we shall derive generating relations for multiple hypergeometric functions of four variables introduced by Exton's [5, 6, 7]. Applying same technique we can also obtain generating relations for hypergeometric functions of four variables recently introduced by Sharma and Parihar [9]. (After excluding those 19 functions, which had already been introduced by Exton [5, 6, 7]).

6.2 GENERATING RELATIONS. In this section, we shall derive some interesting generating relations involving multiple hypergeometric functions of four variables $K_1,\dots, K_{21}$ of Exton [5, 6, 7].

Consider

$$(1-t)^{-a}\ K_1\left(a,a,a,a;b,b,b,c;d,e_1,e_2,d;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p}(c)_q}{(d)_{m+q}(e_1)_n(e_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}(1-t)^{-(a+m+n+p+q)}$$

$$=\sum_{m,n,p,q,r=0}^{\infty}\frac{(a)_{m+n+p+q+r}(b)_{m+n+p}(c)_q}{(d)_{m+q}(e_1)_n(e_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}\frac{t^r}{r!}$$

$$=\sum_{r,m,n,p,q=0}^{\infty}\frac{(a)_r t^r}{r!}\frac{(a+r)_{m+n+p+q}(b)_{m+n+p}(c)_q}{(d)_{m+q}(e_1)_n(e_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

Therefore, we establish

(6.2.1) $$(1-t)^{-a}\ K_1\left(a,a,a,a;b,b,b,c;d,e_1,e_2,d;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{t^r}{r!}K_1\left(a+r,a+r,a+r,a+r;b,b,b,c;d,e_1,e_2,d;x,y,z,u\right).$$

Similarly, applying the same techniques, we also obtained the following generatingrelations :

(6.2.2) $$(1-t)^{b}\ K_1\left(a,a,a,a;b,b,b,c;d,e_1,e_2,d;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},u\right)$$

A paper from this chapter entitled "Generating relations involving hypergeometric functions of four variables", has been published in Pure Appl. Math. Sci., 34 (1991), 15-25.

$$= \sum_{r=0}^{\infty} (b)_r \frac{t^r}{r!} K_1\left(a, a, a, a; b+r, b+r, b+r, c; d, e_1, e_2, d; x, y, z, u\right),$$

(6.2.3) $$(1-t)^{-c} K_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (c)_r \frac{t^r}{r!} K_1\left(a, a, a, a; b, b, b, c+r; d, e_1, e_2, d; x, y, z, u\right),$$

(6.2.4) $$(1-t)^{-a} K_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r!} K_2\left(a+r, a+r, a+r, a+r; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, u\right),$$

(6.2.5) $$(1-t)^{-b} K_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} (b)_r \frac{t^r}{r!} K_2\left(a, a, a, a; b+r, b+r, b+r, c; d_1, d_2, d_3, d_4; x, y, z, u\right),$$

(6.2.6) $$(1-t)^{-c} K_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (c)_r \frac{t^r}{r!} K_2\left(a, a, a, a; b, b, b, c+r; d_1, d_2, d_3, d_4; x, y, z, u\right),$$

(6.2.7) $$(1-t)^{-a} K_3\left(a, a, a, a; b_1, b_1, b_2, b_2, c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r!} K_3\left(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u\right),$$

(6.2.8) $$(1-t)^{-b_1} K_3\left(a, a, a, a; b_1, b_1, b_2, b_2, c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!} t^r K_3\left(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u\right),$$

(6.2.9) $$(1-t)^{-b_2} K_3\left(a, a, a, a; b_1, b_1, b_2, b_2, c_1, c_2, c_2, c_1; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_2)_r}{r!} t^r K_3\left(a, a, a, a; b_1, b_1, b_2+r, b_2+r; c_1, c_2, c_2, c_1; x, y, z, u\right),$$

(6.2.10) $$(1-t)^{-a} K_4\left(a, a, a, a; b_1, b_1, b_2, b_2, c, d_1, d_2, c; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_4\left(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c, d_1, d_2, c; x, y, z, u\right),$$

$$(6.2.11)\quad (1-t)^{-b_1}\, K_4\left(a,a,a,a\,;b_1,b_1,b_2,b_2,c,d_1,d_2,c\,;\frac{x}{1-t},\frac{y}{1-t},z,u\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b_1)_r}{r!}\,t^r\,K_4\left(a,a,a,a\,;b_1+r,b_1+r,b_2,b_2\,;c,d_1,d_2,c\,;x,y,z,u\right),$$

$$(6.2.12)\quad (1-t)^{-b_2}\, K_4\left(a,a,a,a\,;b_1,b_1,b_2,b_2,c,d_1,d_2,c\,;x,y,\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b_2)_r}{r!}\,t^r\,K_4\left(a,a,a,a\,;b_1,b_1,b_2+r,b_2+r\,;c,d_1,d_2,c\,;x,y,z,u\right),$$

$$(6.2.13)\quad (1-t)^{-a}\, K_5\left(a,a,a,a\,;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4\,;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\,t^r\,K_5\left(a+r,a+r,a+r,a+r\,;b_1,b_1,b_2,b_2\,;c_1,c_2,c_3,c_4\,;x,y,z,u\right),$$

$$(6.2.14)\quad (1-t)^{-b_1}\, K_5\left(a,a,a,a\,;b_1,b_1,b_2,b_2,c_1,c_2,c_3,c_4\,;\frac{x}{1-t},\frac{y}{1-t},z,u\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b_1)_r}{r!}\,t^r\,K_5\left(a,a,a,a\,;b_1+r,b_1+r,b_2,b_2\,;c_1,c_2,c_3,c_4\,;x,y,z,u\right),$$

$$(6.2.15)\quad (1-t)^{-b_2}\, K_5\left(a,a,a,a\,;b_1,b_1,b_2,b_2,c_1,c_2,c_3,c_4\,;x,y,\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b_2)_r}{r!}\,t^r\,K_5\left(a,a,a,a\,;b_1,b_1,b_2+r,b_2+r\,;c_1,c_2,c_3,c_4\,;x,y,z,u\right),$$

$$(6.2.16)\quad (1-t)^{-a}\, K_6\left(a,a,a,a\,;b,b,c_1,c_2\,;e,d,d,d\,;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\,t^r\,K_6\left(a+r,a+r,a+r,a+r\,;b,b,c_1,c_2\,;e,d,d,d\,;x,y,z,u\right),$$

$$(6.2.17)\quad (1-t)^{-b}\, K_6\left(a,a,a,a\,;b,b,c_1,c_2\,;e,d,d,d\,;\frac{x}{1-t},\frac{y}{1-t},z,u\right)$$

$$=\sum_{r=0}^{\infty}\frac{(b)_r}{r!}\,t^r\,K_6\left(a,a,a,a\,;b+r,b+r,c_1,c_2\,;e,d,d,d\,;x,y,z,u\right),$$

$$(6.2.18)\quad (1-t)^{-c_1}\, K_6\left(a,a,a,a\,;b,b,c_1,c_2\,;e,d,d,d\,;x,y,\frac{z}{1-t},u\right)$$

$$=\sum_{r=0}^{\infty}\frac{(c_1)_r}{r!}\,t^r\,K_6\left(a,a,a,a\,;b,b,c_1+r,c_2\,;e,d,d,d\,;x,y,z,u\right),$$

$$(6.2.19)\quad (1-t)^{-a}\, K_7\left(a,a,a,a\,;b,b,c_1,c_2\,;d_1,d_2,d_1,d_2\,;\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t},\frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_7\left(a+r, a+r, a+r, a+r; b, b, c_1, c_2; d_1, d_2, d_1, d_2; x, y, z, u\right),$$

(6.2.20) $$(1-t)^{-b} K_7\left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!} t^r K_7\left(a, a, a, a; b+r, b+r, c_1, c_2; d_1, d_2, d_1, d_2; x, y, z, u\right),$$

(6.2.21) $$(1-t)^{-c_1} K_7\left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r!} t^r K_7\left(a, a, a, a; b, b, c_1+r, c_2; d_1, d_2, d_1, d_2; x, y, z, u\right),$$

(6.2.22) $$(1-t)^{-a} K_8\left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_8\left(a+r, a+r, a+r, a+r; b, b, c_1, c_2; d, e_1, d, e_2; x, y, z, u\right),$$

(6.2.23) $$(1-t)^{-b} K_8\left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!} t^r K_8\left(a, a, a, a; b+r, b+r, c_1, c_2; d, e_1, d, e_2; x, y, z, u\right),$$

(6.2.24) $$(1-t)^{-c_1} K_8\left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r!} t^r K_8\left(a, a, a, a; b, b, c_1+r, c_2; d, e_1, d, e_2; x, y, z, u\right),$$

(6.2.25) $$(1-t)^{-a} K_9\left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_9\left(a+r, a+r, a+r, a+r; b, b, c_1, c_2; e_1, e_2, d, d; x, y, z, u\right),$$

(6.2.26) $$(1-t)^{-b} K_9\left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!} t^r K_9\left(a, a, a, a; b+r, b+r, c_1, c_2; e_1, e_2, d, d; x, y, z, u\right),$$

(6.2.27) $$(1-t)^{-c_1} K_9\left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r!} t^r K_9\left(a, a, a, a; b, b, c_1+r, c_2; e_1, e_2, d, d; x, y, z, u\right),$$

$$(6.2.28)\quad (1-t)^{-a}\, K_{10}\left(a, a, a, a\,;\, b, b, c_1, c_2\,;\, d_1, d_2, d_3, d_4\,;\, \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\, K_{10}\left(a+r, a+r, a+r, a+r\,;\, b, b, c_1, c_2\,;\, d_1, d_2, d_3, d_4\,;\, x, y, z, u\right),$$

$$(6.2.29)\quad (1-t)^{-b}\, K_{10}\left(a, a, a, a\,;\, b, b, c_1, c_2\,;\, d_1, d_2, d_3, d_4\,;\, \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!}\, t^r\, K_{10}\left(a, a, a, a\,;\, b+r, b+r, c_1, c_2\,;\, d_1, d_2, d_3, d_4\,;\, x, y, z, u\right),$$

$$(6.2.30)\quad (1-t)^{-c_1}\, K_{10}\left(a, a, a, a\,;\, b, b, c_1, c_2\,;\, d_1, d_2, d_3, d_4\,;\, x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r!}\, t^r\, K_{10}\left(a, a, a, a\,;\, b, b, c_1+r, c_2\,;\, d_1, d_2, d_3, d_4\,;\, x, y, z, u\right),$$

$$(6.2.31)\quad (1-t)^{-a}\, K_{11}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c, c, c, d\,;\, \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\, K_{11}\left(a+r, a+r, a+r, a+r\,;\, b_1, b_2, b_3, b_4\,;\, c, c, c, d\,;\, x, y, z, u\right),$$

$$(6.2.32)\quad (1-t)^{-b_1}\, K_{11}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c, c, c, d\,;\, \frac{x}{1-t}, y, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!}\, t^r\, K_{11}\left(a, a, a, a\,;\, b_1+r, b_2, b_3, b_4\,;\, c, c, c, d\,;\, x, y, z, u\right),$$

$$(6.2.33)\quad (1-t)^{-a}\, K_{12}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c_1, c_1, c_2, c_2\,;\, \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\, K_{12}\left(a+r, a+r, a+r, a+r\,;\, b_1, b_2, b_3, b_4\,;\, c_1, c_1, c_2, c_2\,;\, x, y, z, u\right),$$

$$(6.2.34)\quad (1-t)^{-b_1}\, K_{12}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c_1, c_1, c_2, c_2\,;\, \frac{x}{1-t}, y, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!}\, t^r\, K_{12}\left(a, a, a, a\,;\, b_1+r, b_2, b_3, b_4\,;\, c_1, c_1, c_2, c_2\,;\, x, y, z, u\right),$$

$$(6.2.35)\quad (1-t)^{-a}\, K_{13}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c, c, d_1, d_2\,;\, \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\, K_{13}\left(a+r, a+r, a+r, a+r\,;\, b_1, b_2, b_3, b_4\,;\, c, c, d_1, d_2\,;\, x, y, z, u\right),$$

$$(6.2.36)\quad (1-t)^{-b_1}\, K_{13}\left(a, a, a, a\,;\, b_1, b_2, b_3, b_4\,;\, c, c, d_1, d_2\,;\, \frac{x}{1-t}, y, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!} t^r K_{13}\left(a, a, a, a ; b_1 + r, b_2, b_3, b_4 ; c, c, d_1, d_2 ; x, y, z, u\right),$$

(6.2.37) $$(1-t)^{-a} K_{14}\left(a, a, a, c_3 ; b, c_1, c_2, b ; d, d, d, d ; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_{14}\left(a+r, a+r, a+r, c_3 ; b, c_1, c_2, b ; d, d, d, d ; x, y, z, u\right),$$

(6.2.38) $$(1-t)^{-c_3} K_{14}\left(a, a, a, c_3 ; b, c_1, c_2, b ; d, d, d, d ; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_3)_r}{r!} t^r K_{14}\left(a, a, a, c_3 + r ; b, c_1, c_2, b ; d, d, d, d ; x, y, z, u\right),$$

(6.2.39) $$(1-t)^{-b} K_{14}\left(a, a, a, c_3 ; b, c_1, c_2, b ; d, d, d, d ; \frac{x}{1-t}, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!} t^r K_{14}\left(a, a, a, c_3 ; b+r, c_1, c_2, b+r ; d, d, d, d ; x, y, z, u\right),$$

(6.2.40) $$(1-t)^{-c_1} K_{14}\left(a, a, a, c_3 ; b, c_1, c_2, b ; d, d, d, d ; x, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r!} t^r K_{14}\left(a, a, a, c_3 ; b, c_1 + r, c_2, b ; d, d, d, d ; x, y, z, u\right),$$

(6.2.41) $$(1-t)^{-a} K_{15}\left(a, a, a, b_5 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_{15}\left(a+r, a+r, a+r, b_5 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x, y, z, u\right),$$

(6.2.42) $$(1-t)^{-b_5} K_{15}\left(a, a, a, b_5 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_5)_r}{r!} t^r K_{15}\left(a, a, a, b_5 + r ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x, y, z, u\right),$$

(6.2.43) $$(1-t)^{-b_1} K_{15}\left(a, a, a, b_5 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; \frac{x}{1-t}, y, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!} t^r K_{15}\left(a, a, a, b_5 ; b_1 + r, b_2, b_3, b_4 ; c, c, c, c ; x, y, z, u\right),$$

(6.2.44) $$(1-t)^{-a_1} K_{16}\left(a_1, a_2, a_3, a_4 ; b ; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r}{r!} t^r K_{16}\left(a_1 + r, a_2, a_3, a_4 ; b ; x, y, z, u\right),$$

$$(6.2.45)\quad (1-t)^{-a_2}\, K_{16}\left(a_1, a_2, a_3, a_4 ; b ; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_2)_r}{r!}\, t^r\, K_{16}\left(a_1, a_2+r, a_3, a_4 ; b ; x, y, z, u\right),$$

$$(6.2.46)\quad (1-t)^{-a_3}\, K_{16}\left(a_1, a_2, a_3, a_4 ; b ; x, \frac{y}{1-t}, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_3)_r}{r!}\, t^r\, K_{16}\left(a_1, a_2, a_3+r, a_4 ; b ; x, y, z, u\right),$$

$$(6.2.47)\quad (1-t)^{-a_4}\, K_{16}\left(a_1, a_2, a_3, a_4 ; b ; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_4)_r}{r!}\, t^r\, K_{16}\left(a_1, a_2, a_3, a_4+r ; b ; x, y, z, u\right),$$

$$(6.2.48)\quad (1-t)^{-a_1}\, K_{17}\left(a_1, a_2, a_3 ; b_1, b_2 ; c ; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r}{r!}\, t^r\, K_{17}\left(a_1+r, a_2, a_3, b_1, b_2 ; c ; x, y, z, u\right),$$

$$(6.2.49)\quad (1-t)^{-a_2}\, K_{17}\left(a_1, a_2, a_3, b_1, b_2 ; c ; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_2)_r}{r!}\, t^r\, K_{17}\left(a_1, a_2+r, a_3, b_1, b_2 ; c ; x, y, z, u\right),$$

$$(6.2.50)\quad (1-t)^{-a_3}\, K_{17}\left(a_1, a_2, a_3, b_1, b_2 ; c ; x, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_3)_r}{r!}\, t^r\, K_{17}\left(a_1, a_2, a_3+r, b_1, b_2 ; c ; x, y, z, u\right),$$

$$(6.2.51)\quad (1-t)^{-b_1}\, K_{17}\left(a_1, a_2, a_3, b_1, b_2 ; c ; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!}\, t^r\, K_{17}\left(a_1, a_2, a_3, b_1+r, b_2 ; c ; x, y, z, u\right),$$

$$(6.2.52)\quad (1-t)^{-a_1}\, K_{18}\left(a_1, a_2, a_3, b_1, b_2 ; c ; \frac{x}{1-t}, \frac{y}{1-t}, z. u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r}{r!}\, t^r\, K_{18}\left(a_1+r, a_2, a_3, b_1, b_2 ; c ; x, y, z, u\right),$$

$$(6.2.53)\quad (1-t)^{-a_2}\, K_{18}\left(a_1, a_2, a_3, b_1, b_2 ; c ; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_2)_r}{r!} t^r K_{18}\left(a_1, a_2+r, a_3, b_1, b_2; c; x, y, z, u\right),$$

(6.2.54) $$(1-t)^{-a_3} K_{18}\left(a_1, a_2, a_3, b_1, b_2; c; x, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_3)_r}{r!} t^r K_{18}\left(a_1, a_2, a_3+r, b_1, b_2; c; x, y, z, u\right),$$

(6.2.55) $$(1-t)^{-b_1} K_{18}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!} t^r K_{18}\left(a_1, a_2, a_3, b_1+r, b_2; c; x, y, z, u\right),$$

(6.2.56) $$(1-t)^{-a_1} K_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r}{r!} t^r K_{19}\left(a_1+r, a_2, b_1, b_2, b_3, b_4; c; x, y, z, u\right),$$

(6.2.57) $$(1-t)^{-a_2} K_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_2)_r}{r!} t^r K_{19}\left(a_1, a_2+r, b_1, b_2, b_3, b_4; c; x, y, z, u\right),$$

(6.2.58) $$(1-t)^{-b_1} K_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; x, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r!} t^r K_{19}\left(a_1, a_2, b_1+r, b_2, b_3, b_4; c; x, y, z, u\right),$$

(6.2.59) $$(1-t)^{-a_1} K_{20}\left(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r}{r!} t^r K_{20}\left(a_1+r, a_1+r, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; x, y, z, u\right),$$

(6.2.60) $$(1-t)^{-a_2} K_{20}\left(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_2)_r}{r!} t^r K_{20}\left(a_1, a_1, b_3, b_4; b_1, b_2, a_2+r, a_2+r; c, c, c, c; x, y, z, u\right),$$

(6.2.61) $$(1-t)^{-a} K_{21}\left(a, a, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!} t^r K_{21}\left(a+r, a+r, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u\right),$$

and

$$(6.2.62)\quad (1-t)^{-b_5}\, K_{21}\left(a, a, b_5, b_6 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_5)_r}{r!}\, t^r\, K_{21}\left(a, a, b_5+r, b_6 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x, y, z, u\right),$$

Applying the same techniques, we can also obtain generating relations for the hypergeometric functions of four variables recently introduced by Sharma and Parihar [9]. (After excluding those 19 functions which had already been introduced by Exton [5, 6, 7]).

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CHAPTER - VII

A MULTILINEAR GENERATING FUNCTION

7.1 INTRODUCTION.

Recently, Srivastava [19] developed a fairly elementary method of proving a general multilinear generating function involving the Srivastava- Singhal polynomials $G_n^{(\alpha)}$ (x, h, p, k) defined by Rodrigues' formula [17, p.75, eq.(1.3)]

(7.1.1) $$G_n^{(\alpha)}(x, h, p, k) = \frac{x^{-kn-\alpha}}{n!} \exp(p x^h)(x^{k+1} D_x)^n \{x^\alpha \exp(-p x^h)\}, \; D_x = \frac{d}{dx}$$

which upon suitable specializations, yields a number of interesting results including, for example a multivariable hypergeometric generating function for the biorthogonal polynomials sets

$\{Y_n^\alpha (x ; k)\}$ and Z_n^α (x; k) introduced by Konhauser [10, 11].

In this chapter, we shall obtain a general multilinear generating function involving a general class of polynomials $\{G_n$ (h, g, k)$\}$ introduced by Chandel [4, p.45, eq. (1.4)]

(7.1.2) $$G_n(h, g, k) = e^{-hg} \Omega_x^n e^{hg}, \; \Omega_x = x^k \frac{d}{dx}, \; k \neq 1,$$

where h, k are independent of x and g is any differentiable function of x. Finally, we shall also discuss various special cases of main result as its applications.

For k = 0, (7.1.2) reduces to Bell polynomials [12] defined by

(7.1.3) $$H_n(g, h) = (-1)^n e^{-hg} D_x^n e^{hg},$$

while when $k \to 1$, (7.1.2) includes Srivastava's polynomials [15] defined by

(7.1.4) $$G_n(h, g) = e^{-hg} (x \frac{d}{dx})^n e^{hg},$$

as a special case.

For h = 1, g(x) = α log x $-$ px^h and replacing k by k + 1, (7.1.2) includes (7.1.1) in the following way:

(7.1.5) $$G_n(1, \alpha \log x - px^h, k+1) = n! \, x^{kn} \, G_n^{(\alpha)}(x, h, p, k),$$

while for h = 1, g(x) = α log x $-$ px^r, (7.1.2) reduces to Chandel polynomials ([2], [3]) defined by

(7.1.6) $$T_n^{\alpha, k}(x, r, p) = x^{-\alpha} e^{px^r} \Omega_x^n \{x^\alpha e^{-px^r}\}.$$

7.2 MAIN RESULT.

Our main result is contained in the following Theorem. For a bounded multiple sequence $\{\Lambda(n_1,..., n_r)\}$, let

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A paper from this chapter, entitled "A Multilinear Generating Function" has been accepted for publication in Mathematics Education.

$$(7.2.1)\quad H(n_1,\dots,n_r,y_1,\dots,y_r) = \sum_{j_1=0}^{[n_1/m_1]} \cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 j_1}}{j_1!} \cdots \frac{(-n_r)_{m_r j_r}}{j_r!} \Lambda (j_1,\dots,j_r)\, y_1^{j_1},\dots,y_r^{j_r},$$

where $m_1,\dots,m_r$, are positive integers and $r = 1, 2, 3,\dots$ then for every non-negative integer m

$$(7.2.2)\quad \sum_{n_1,\dots,n_r=0}^{\infty} G_{m+n_1+\dots+n_r}(h,g,k)\; H(n_1,\dots,n_r;y_1,\dots,y_r)\,\frac{u_1^{n_1}}{n_1!}\cdots\frac{u_r^{n_r}}{n_r!}$$
$$= \exp\{-h\,g(x) + (u_1+\dots+u_r)\,\Omega_x\}$$
$$\sum_{n_1,\dots,n_r=0}^{\infty} \Lambda(n_1,\dots,n_r) \prod_{i=1}^{r} \frac{\left((-1)^{m_i} u_i^{m_i} y_i\right)^{n_i}}{n_i!}$$
$$\Omega_x^{m+m_1 n_1+\dots+m_r n_r}\{\exp(h\,g(x)\},$$

provided that the multiple series on the right-hand side of (7.2.2) has a meaning, and $k \neq 1$.

7.3 PROOF OF THE THEOREM.

For convenience, let $\Omega(u_1,\dots,u_r)$ denotes the left-hand side of (7.2.2) and for brevity take

$$(7.3.1)\quad N = n_1+\dots+n_r \text{ and } J = m_1 j_1+\dots+m_r j_r.$$

Now making an appeal to the following explicit formula due to Chandel [4, (5.1)]:

$$(7.3.2)\qquad G_n(h,g,k) = \sum_{s=0}^{n} \frac{(-1)^s h^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} [g(x)]^{s-j}\, \Omega_x^n\, [g(x)]^j,$$

and identify due to Srivastava [18, p.4, eq, 12]

$$(7.3.3)\qquad \sum_{n_1,\dots,n_r=0}^{\infty} f(n_1+\dots+n_r)\,\frac{u_1^{n_1}}{n_1!}\cdots\frac{u_r^{n_r}}{n_r!} = \sum_{n=0}^{\infty} f(n)\,\frac{(u_1+\dots+u_r)^n}{n!},$$

we have

$$\Omega(u_1,\dots,u_r) = \sum_{n_1,\dots,n_r=0}^{\infty} \frac{u_1^{n_1}}{n_1!}\cdots\frac{u_r^{n_r}}{n_r!}$$
$$\sum_{j_1=0}^{[n_1/m_1]} \cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 j_1}}{j_1!} \cdots \frac{(-n_r)_{m_r j_r}}{j_r!} \Lambda(j_1,\dots,j_r)\, y_1^{j_1}\cdots y_r^{j_r}$$
$$\sum_{s=0}^{m+N} \frac{(-1)^s h^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} [g(x)]^{s-j}\, \Omega_x^{m+N}\, [g(x)]^j$$
$$= \sum_{j_1,\dots,j_r=0}^{\infty} \Lambda(j_1,\dots,j_r) \frac{\left[(-1)^{m_1} u_1^{m_1} y_1\right]^{j_1}}{j_1!} \cdots \frac{\left[(-1)^{m_r} u_r^{m_r} y_r\right]^{j_r}}{j_r!}$$

$$\sum_{n=0}^{\infty} \frac{(u_1+\ldots+u_r)^n}{n!} \sum_{s=0}^{m+n+J} \frac{(-1)^s h^s}{s!} \sum_{j=0}^{\infty} (-1)^j \binom{s}{j} [g(x)]^{s-j}\, \Omega_x^{m+n+J} [g(x)]^j$$

$$= \sum_{n, j_1,\ldots,j_r=0}^{\infty} \Lambda(j_1,\ldots,j_r) \frac{(u_1+\ldots+u_r)^n}{n!} \frac{\left[(-1)^{m_1} u_1^{m_1} y_1\right]^{j_1}}{j_1!} \cdots \frac{\left[(-1)^{m_r} u_r^{m_r} y_r\right]^{j_r}}{j_r!}$$

$$\sum_{s=0}^{m+n+J} \frac{(-1)^s h^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} [g(x)]^{s-j}\, \Omega_x^{m+n+J} [g(x)]^j.$$

The innermost some in the above expression being the s^{th} difference of a polynomial of degree $m + n + J$, is nil when $s > m + n + J$, therefore, finally we derive (7.2.2) under the condition of (7.2.3).

7.4 APPLICATIONS.

Specialising the values of the arbitrary coefficients $\Lambda(j_1,\ldots, j_r)$ in (7.2.1), we can obtain the following result involving multiple hypergeometric function of Srivastava and Daoust [16] :

$$(7.4.1) \qquad \sum_{n_1,\ldots,n_r=0}^{\infty} G_{m+n_1+\ldots+n_r}(h, g, k)$$

$$F^{A:B'+1;\ldots;B^{(r)}+1}_{C:D';\ldots;D^{(r)}} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[-n_1:m_1],[(b'):\Phi'];\ldots; \\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} [-n_r:m_r],[(b^{(r)}):\Phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}]; \end{matrix} \; y_1,\ldots,y_r \right) , \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_r^{n_r}}{n_r!}$$

$$= \exp[-h\, g(x) + (u_1+\ldots+u_r)\,\Omega_x]$$

$$F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}} \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] :[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}]; \\ [(c):\psi,\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \right.$$

$$\left. (-u_1\Omega_x)^{m_1} y_1,\ldots,(-u_r\Omega x)^{m_r} y_r \right) \Omega_x^m \{\exp(h\, g(x))\},\; k \neq 1.$$

As an application of main theorem, we also obtain

$$(7.4.2) \qquad \sum_{n_1,\ldots,n_r=0}^{\infty} G_{m+n_1+\ldots+n_r}(h, g(x), k)$$

$$\prod_{i=1}^{r} {}_{m_i+B^{(i)}}F_{D^{(i)}} \left[\begin{matrix} \Delta(m_i,-n_i),(b^{(i)}); \\ (d^{(i)}); \end{matrix} \; y_i\, m_i^{m_i} \right] \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_r^{n_r}}{n_r!}$$

$$= \exp[-h\, g(x) + (u_1+\ldots+u_r)\,\Omega_x]$$

$$\prod_{i=1}^{r} {}_{B^{(i)}}F_{D^{(i)}} \left[\begin{matrix} (b^{(i)}); \\ (d^{(i)}); \end{matrix} \; y_i(-u_i)^{m_i}\Omega_x^{m_i} \right] \Omega_x^m \; [\exp(h\, g(x))],\; k \neq 1.$$

7.5 SPECIAL CASES.

Since the polynomials defined by (7.1.2) are generalization of Laguerre, Hermite and Bessel polynomials, Truesdell polynomials [8], Bell polynomials [12] and the polynomials studied by Chandel [2, 3], Chatterjea [5, 6, 7], Chak [1], Gould and Hopper [9], Singh [13], Singh-Srivastava [14], Srivastava [15] and Srivastava - Singhal [17], therefore, specializing the values of arbitrary coefficients $\Lambda(j_1,\ldots, j_r)$ in (7.2.1) and (7.2.2), we can obtain several results involving these polynomials and the multiple hypergeometric function of Srivastava and daoust [16].

For $h = 1$, $g(x) = \alpha \log x - px^r$. (4.1) gives the following result for Chandel polynomials $T_n^{(\alpha, k)}(x, r, p)$ defined by (7.1.6):

$$(7.5.1) \qquad \sum_{n_1, \ldots, n_r = 0}^{\infty} T_{m + n_1 + \ldots + n_r}^{(\alpha, k+1)}(x, r, p)$$

$$F^{A : B'+1 ; \ldots ; B^{(r)}+1}_{C : D' ; \ldots ; D^{(r)}}\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [-n_1 : m_1], [(b') : \Phi'] ; \; [-n_r : m_r] ; [(b^{(r)}) : \Phi^{(r)}] ; \\ [(c) : \psi', \ldots, \psi^{(r)}] : [(d') : \delta'] ; \ldots ; \qquad [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \; y_1, \ldots, y_r\right)$$

$$\frac{\left(\frac{u_1}{k x^k}\right)^{n_1}}{n_1 !}, \ldots, \frac{\left(\frac{u_r}{k x^k}\right)^{n_r}}{n_r !}$$

$$= e^{p x^r} k^m x^{mk} \left(\frac{\alpha}{k}\right)_m \Delta_r^{-m - \alpha/k}$$

$$F^{A+1 : O ; B' ; \ldots ; B^{(r)}}_{C : 1 ; D' ; \ldots ; D^{(r)}}\left(\begin{matrix} [(a) : 0, \theta', \ldots, \theta^{(r)}] & \left[\frac{\alpha}{k} + m : \frac{r}{k}, m_1, \ldots, m_r\right] : \\ [(c) : 0, \psi', \ldots, \psi^{(r)}] : & \left[\frac{\alpha}{k} : \frac{r}{k}\right] ; \end{matrix}\right.$$

$$\left.\begin{matrix} - ; [(b') : \Phi'] ; \ldots ; [(b^{(r)}) : \Phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} \; \frac{-p x^r}{\Delta_r^{r/k}}, \; y_1 \left(\frac{-u_1}{x^k \Delta_r}\right)^{m_1}, \ldots, y_r \left(\frac{-u_r}{\Delta_r x^k}\right)^{m_r}\right)$$

where $\Delta_r = 1 - (u_1 + \ldots + u_r)$, $|u_1 + \ldots + u_r| < 1$, and $k \neq 0$.

Similarly for $h = 1$, $g(x) = \alpha \log x - p x^r$, (7.4.2) gives

$$(7.5.2) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} T_{m + n_1 + \ldots + n_r}^{\alpha, k}(x, r, p)$$

$$\prod_{i=1}^{r} {}_{m_i + B^{(i)}} F_{D^{(i)}} \left[\begin{matrix} \Delta(m_i, -n_i), (b^{(i)}) ; \\ (d^{(i)}) ; \end{matrix} \; y_i \, m_i\right] \frac{u_1^{n_1}}{n_1 !} \cdots \frac{u_r^{n_r}}{n_r !}$$

$$= x^{-\alpha} \exp(p x^r) (k-1)^m \left(\frac{\alpha}{k-1}\right)_m \Delta_{k,x}^{\alpha + m}$$

$$F^{1 : O ; B' ; \ldots ; B^{(r)}}_{0 : 1 ; D' ; \ldots ; D^{(r)}}\left(\begin{matrix} \left[\frac{\alpha}{k-1} + m : \frac{r}{k-1}, m_1, \ldots, m_r\right] : - ; & [(b') : 1] ; \ldots ; [(b^{(r)}) : 1] ; \\ - ; \left[\frac{\alpha}{k-1} : \frac{r}{k-1}\right] ; [(d') : 1] ; \ldots ; & [(d^{(r)}) : 1] ; \end{matrix}\right.$$

$$\left. -p \Delta_{k,x}^{r}, \; y_1 (k-1)^{m_1} \Delta_{k,x}^{(k-1) m_1}, \ldots, y_r (k-1)^{m_r} \Delta_{k,x}^{(k-1) m_r}\right)$$

where $\Delta_{k,x} = \dfrac{x}{\left(1 - (k-1)(u_1 + \ldots + u_r) x^{k-1}\right)^{1/(k-1)}}$,

$|(k-1)(u_1 + \ldots + u_r) x^{k-1}| < 1$ and $k \neq 1$.

7.6 APPLICATIONS OF THE MAIN THEOREM TO SRIVASTAVA-SINGHAL POLYNOMIALS.

For particular interest, choosing $h = 1$, $g(x) = \alpha \log x - px^h$ and replacing k by $k+1$ in our main theorem, we derive the following

Theorem 2. For a bounded multiple sequence $\{\Lambda(n_1, \ldots, n_r)\}$, let

(7.6.1)

$$H(n_1, \ldots, n_r ; y_1, \ldots, y_r) = \sum_{j_1=0}^{[n_1/m_1]} \cdots \sum_{j_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 j_1}}{j_1 !} \cdots \frac{(-n_r)_{m_r j_r}}{j_r !} \Lambda(j_1, \ldots, j_r)\, y_1^{j_1} \cdots y_r^{j_r},$$

where $m_1, \ldots, m_r$ are positive integers and $r = 1, 2, 3, \ldots$ then for every non-negative integer m

$$(7.6.2) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \ldots + n_r)!\, x^{k(m + n_1 + \ldots + n_r)}$$

$$G^{(\alpha)}_{m + n_1 + \ldots + n_r}(x, h, p, k)\, H_{(n_1, \ldots, n_r ; y_1, \ldots, y_r)} \frac{u_1^{n_1}}{n_1 !} \cdots \frac{u_r^{n_r}}{n_r !}$$

$$= x^{-\alpha} \exp(px^h)\, k^m\, \Delta^{\alpha + mk}_{k+1, x}$$

$$\sum_{n, n_1, \ldots, n_r = 0}^{\infty} \left(\frac{\alpha + nh}{k}\right)_{m + m_1 n_1 + \ldots + m_r n_r} \Lambda(n_1, \ldots, n_r)$$

$$\frac{\left(-p\, \Delta^{h}_{k+1, x}\right)^n}{n!} \prod_{i=1}^{r} \frac{\left[(-1)^{m_i} y_i u_i^{m_i} k^{m_i} \Delta^{k m_i}_{k+1, x}\right]^{n_i}}{n_i !}$$

where $|k(u_1 + \ldots + u_r) x^k| < 1$ and $k \neq 0$.

This theorem is quite different from the main theorem of Srivastava [19, p.185].

Specializing the parameters $\Lambda(n_1, \ldots, n_r)$ in theorem 2, we derive the following multilinear generating relation for the polynomials $G^{(\alpha)}_n(x, h, p, k)$ of Srivastava-Singhal [17] :

$$(7.6.3) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (m + n_1 + \ldots + n_r)!\; x^{k(m + n_1 + \ldots + n_r)}\, G^{(\alpha)}_{m + n_1 + \ldots + n_r}(x,h,p,k)$$

$$F^{A : B' + 1 ; \ldots ; B^{(r)} + 1}_{C : D' ; \ldots ; D^{(r)}} \left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [-n_1, m_1], [(b') : \Phi'] ; \ldots ; \\ [(c) : \psi', \ldots, \psi^{(r)}] : [(d') : \delta'] ; \ldots ; \end{matrix} \right.$$

$$\left. \begin{matrix} [-n_r : m_r], \left[(b^{(r)}) : \Phi^{(r)}\right] ; \\ \left[(d^{(r)}) : \delta^{(r)}\right] ; \end{matrix} \; y_1, \ldots, y_r \right) \frac{u_1^{n_1}}{n_1 !} \cdots \frac{u_r^{n_r}}{n_r !}$$

$$= x^{-\alpha} \exp(px^h)\, k^m \left(\frac{\alpha}{k}\right)_m \Delta^{\alpha + m}_{k+1, x}$$

$$F^{A + 1 : 0 ; B' ; \ldots ; B^{(r)}}_{C : 1 ; D' ; \ldots ; D^{(r)}} \left(\begin{matrix} [(a) : 0, \theta', \ldots, \theta^{(r)}], \left[\frac{\alpha}{k} + m : \frac{h}{k}, m_1, \ldots, m_r\right] : \\ [(c) : 0 . \psi', \ldots, \psi^{(r)}] : \left[\frac{\alpha}{k} : \frac{h}{k}\right] ; \end{matrix} \right.$$

$$\begin{matrix} - : [(b') : \Phi'] ; \ldots ; [(b^{(r)}) : \Phi^{(r)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ; \end{matrix}$$

$$\left. -p\, \Delta^{h}_{k+1, x}\; y_1 k^{m_1} \Delta^{k m_1}_{k+1, x}, \ldots, y_r k^{m_r} \Delta^{k m_r}_{k+1, x} \right)$$

where $|k(u_1 + \ldots + u_r) x^k| < 1$ and $k \neq 0$.

which is quite different from the result due to Srivastava [17, p.188, (18)].

Similalrly, we obtain the following result involvingSrivastava and Singhal polynomials

$$(7.6.4)\quad \sum_{n_1,\dots,n_r=0}^{\infty} (m+n_1+\dots+n_r)!\, x^{k(m+n_1+\dots+n_r)}\, \mathbf{G}^{(\alpha)}_{m+n_1+\dots+n_r}(x,h,p,k)$$

$$\prod_{i=1}^{r} {}_{m_i+B^{(i)}}\mathbf{F}_{D^{(i)}}\left[\begin{matrix}\Delta(m_i-n_i),(b^{(i)});\\ (d^{(i)});\end{matrix}\; y_i m_i\right] \frac{u_1^{n_1}}{n_1!}\cdots\frac{u_r^{n_r}}{n_r!}$$

$$= x^{-\alpha}\exp(px^h)\, k^m \left(\frac{\alpha}{k}\right)_m \Delta^{\alpha+m}_{k+1,x}$$

$$\mathbf{F}^{1:0;B';\dots;B^{(r)}}_{0:1;D';\dots;D^{(r)}}\left(\begin{matrix}\left[\frac{\alpha}{k}+m:\frac{r}{k},m_1,\dots,m_r\right]:-\,; & [(b'):1];\dots;[(b^{(r)}):1];\\ -:\left[\frac{\alpha}{k}:\frac{r}{k}\right];[(d'):1];\dots; & [(d^{(r)}):1];\end{matrix}\right.$$

$$\left.-p\Delta^{h}_{k+1,x},\; y_1 k^{m_1}\Delta^{k m_1}_{k+1,x},\dots, y_r k^{m_r}\Delta^{k m_r}_{k+1,x}\right),$$

where $|\, k(u_1+\dots+u_r)x^k \,| < 1$ and $k \neq 0$.

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CHAPTER - VIII

MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST AND ITS APPLICATIONS IN A PROBLEM INVOLVING LAPLACE EQUATION

8.1 INTRODUCTION.

Recently Chandel-Yadava [1] and Chandel-Gupta [2] have made applications of multiple hypergeometric function of several variables of Srivastava and Daoust [4, 5] (Also see Srivastava and Karlsson [6]) in different problems on heat conduction. In this Chapter, first we evaluate an interesting integral involving about multiple hypergeometric function of several variables of Srivastava and Daoust [4, 5, 6]

$$(8.1.1)\quad S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}];[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\ldots,x_m\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma(a_j+\sum_{i=1}^{n}m_i\theta_j^{(i)})\prod_{j=1}^{B}\Gamma(b'_j+m_1\Phi'_j)\ldots\prod_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)}+m_n\Phi_j^{(n)})}{\prod_{j=1}^{C}\Gamma(c_j+\sum_{i=1}^{n}m_i\psi_j^{(i)})\prod_{j=1}^{D}\Gamma(d'_j+m_1\delta'_j)\ldots\prod_{j=1}^{D^{(n)}}\Gamma(d_j^{(n)}+m_n\delta_j^{(n)})}\ \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

where

$$\theta_j^{(i)},\ j=1,\ldots,A;\ \Phi_j^{(i)},\ J=1,\ldots,B^{(i)};\ \psi_j^{(i)},\ j=1,\ldots,C;\ \delta_j^{(i)},\ j=1,\ldots,D^{(i)};\quad 1\le i\le n,$$

are real and positive and (a) is taken to abbreviate the sequence of A parameters $a_1,\ldots,a_A$, $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b_1^{(i)},\ldots,b_{B^{(i)}}^{(i)}$, $i=1,\ldots,n$ with similar interpretation for (c) and $(d^{(i)})$, $i=1,\ldots,n$ etc, and then we make its application to solve a problem on heat conduction involving Laplace equation. Finally, we also derive an expansion formula involving above multiple hypergeometric function.

8.2 INTEGRAL.

In this section, making an appeal to the integral

$$(8.2.1)\quad \int_0^a \cos^m\frac{\pi x}{a}\cos p\frac{\pi x}{a}\,dx=\frac{a}{\sqrt{\pi}\,2^p}\ \frac{\Gamma(m+1)\,\Gamma\left(\frac{m-p+1}{2}\right)}{\Gamma(m-p+1)\,\Gamma\left(\frac{m+p+2}{2}\right)},$$

where m, p are positive integers such that $m>p$ and $m-p$ is even, we evaluate

$$(8.2.2)\quad \int_0^a \cos^{m-1}\frac{\pi x}{a}\cos p\frac{\pi x}{a}\,S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1\cos^{2\sigma_1}\frac{\pi x}{a},\ldots,z_n\cos^{2\sigma_n}\frac{\pi x}{a}\right)dx$$

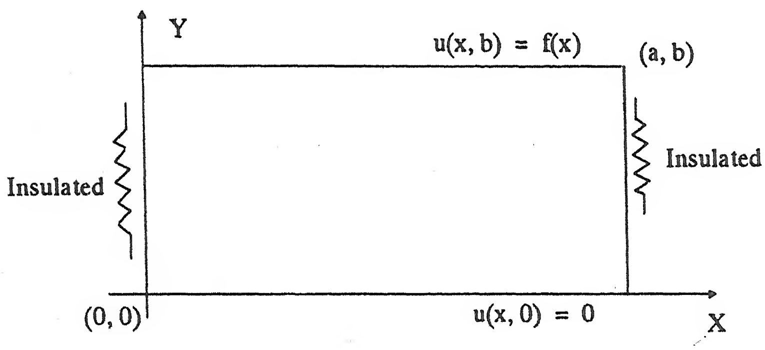
$$=\frac{a}{\sqrt{\pi}\,2^p}\,S^{A+2:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\ [m:2\sigma_1,\ldots,2\sigma_n],\\ [(c):\psi',\ldots,\psi^{(n)}],[(m-p:2\sigma_1,\ldots,2\sigma_n],\end{matrix}\right.$$

$$\left[\frac{m-p}{2} : \sigma_1, \ldots, \sigma_n\right] : [(b') : \Phi'] ; \ldots ; [(b^{(n)}) : \Phi^{(n)}] ; \qquad z_1, \ldots, z_n \Big)$$
$$\left[\frac{m-p+1}{2} : \sigma_1, \ldots, \sigma_n\right] : [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ;$$

provided that m, p are positive integers such that $m > p + 1$ and $(m-p)$ is odd integer; and

$$1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \Phi_j^{(i)} > 0, \; i = 1, \ldots, n.$$

8.3 PROBLEM. We shall find the steady state temperature $u(x, y)$ in a rectangular plate with following boundary conditions when no heat escapes from the lateral faces of the plate



(8.3.1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \; 0 < y < b$$

with the boundary conditions

(8.3.2) $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad 0 < y < b,$$

(8.3.3) $u(x, 0) = 0, \quad 0 < x < a,$

(8.3.4) $u(x, b) = f(x), \; 0 < x < a.$

We shall cosdier the problem of determining $u(x, y)$,

where

(8.3.5) $$u(x, 0) = f(x) = \cos^{m-1}\left(\frac{\pi x}{a}\right) S^{A : B' ; \ldots ; B^{(n)}}_{C : D' ; \ldots ; D^{(n)}} \left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(n)}] : \\ [(c) : \psi', \ldots, \psi^{(n)}] : \end{matrix} \right.$$
$$\left. \begin{matrix} [(b') : \Phi'] ; \ldots ; [(b^{(n)}) : \Phi^{(n)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} \; z_1 \cos^{2\sigma_1} \frac{\pi x}{a}, \ldots, z_n \cos^{2\sigma_n} \frac{\pi x}{a} \right).$$

8.4 SOLUTION OF THE PROBLEM. According to Zill [3, p.468,(10.4.3)]

(8.4.1) $$u(x, y) = A_o y + \sum_{p=1}^{\infty} A_p \sinh \frac{p\pi y}{a} \cos \frac{p\pi x}{a}.$$

For $y = b$

$$(8.4.2)\quad u(x,b) = f(x) = A_o b + \sum_{p=1}^{\infty} A_p \sinh\frac{p\pi b}{a}\cos\frac{p\pi x}{a},$$

which, in this case, is a half range expansion of "f" in a cosine series. If we make the identifications $A_o b = \frac{a_o}{2}$ and $A_p \sinh\frac{p\pi b}{a} = a_n$, n =1, 2, 3,..., it follows from Zill [3, p.449, (10.2.2)]

$$(8.4.3)\quad A_o = \frac{1}{ab}\int_0^a f(x)\,dx$$

and

$$(8.4.4)\quad A_p = \frac{2}{a\sinh\frac{p\pi b}{a}}\int_0^a f(x)\cos\frac{p\pi x}{a}\,dx .$$

Making an appeal to integral (8.2.2), (8.4.4) gives

$$(8.4.5)\quad A_p = \frac{1}{\sqrt{\pi}\,2^{p-1}\sinh\frac{p\pi b}{a}}\, S^{A+2:B';...;B^{(n)}}_{C+2:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}],\\ [(c):\psi',...,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[m:2\sigma_1,...,2\sigma_n],\left[\frac{m-2}{2}:\sigma_1,...,\sigma_n\right]:[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}];\\ [m-p:2\sigma_1,...,2\sigma_n],\left[\frac{m+p+1}{2}:\sigma_1,...,\sigma_n\right]:[(d'):\delta'];...;(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,...,z_n\right),$$

provided all the conditions of (8.2.2) are satisfied.

Now making an appeal to (8.4.3), (8.4.5) and (8.2.2), we derive

$$(8.4.6)\quad A_o = \frac{1}{b\sqrt{\pi}}\, S^{A+1:B';...;B^{(n)}}_{C+1:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}],\\ [(c):\psi',...,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[\frac{m}{2}:\sigma_1,...,\sigma_n]:[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}];\\ [\frac{m+1}{2}:\sigma_1,...,\sigma_n]:[(d'):\delta'];...;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,...,z_n\right).$$

Substituting the values of A_o and A_p in (8.4.1) we get the following required solution of the problem

$$(8.4.7)\quad u(x,y) = \frac{y}{\sqrt{\pi}\,b}\, S^{A+1:B';...;B^{(n)}}_{C+1:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}],\\ [(c):\psi',...,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[\frac{m}{2}:\sigma_1,...,\sigma_n]:[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}];\\ [\frac{m+1}{2}:\sigma_1,...,\sigma_n]:[(d'):\delta'];...;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,...,z_n\right)$$

$$+\sum_{p=1}^{\infty}\frac{\sinh\left(\frac{p\pi y}{a}\right)\cos\left(\frac{p\pi x}{a}\right)}{\sqrt{\pi}\,2^{p-1}\sinh\frac{p\pi b}{a}}\, S^{A+2:B';...;B^{(n)}}_{C+2:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}],\\ [(c):\psi',...,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[m:2\sigma_1,...,2\sigma_n],\left[\frac{m-2}{2}:\sigma_1,...,\sigma_n\right]:[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}];\\ [m-p:2\sigma_1,...,2\sigma_n],\left[\frac{m+p+1}{2}:\sigma_1,...,\sigma_n\right]:[(d'):\delta'];...;(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,...,z_n\right).$$

provided that all the conditions of (8.2.2) are satisfied.

8.5 EXPANSION FORMULA. Making an appeal to (8.3.5), (8.4.2), (8.4.5) and (8.4.6), we derive

$$\text{(8.5.1)} \qquad \cos^{m-1}\frac{\pi x}{x}\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1\cos^{2\sigma_1}\frac{\pi x}{a},\ldots,z_n\cos^{2\sigma_n}\frac{\pi x}{a}\right)$$

$$= \frac{1}{\sqrt{\pi}}\, S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[\frac{m}{2}:\sigma_1,\ldots,\sigma_n]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [\frac{m+1}{2}:\sigma_1,\ldots,\sigma_n]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\ldots,z_n\right)$$

$$+ \frac{1}{\sqrt{\pi}}\sum_{p=1}^{\infty}\frac{\cos p\frac{\pi x}{a}}{2^{p-1}}\, S^{A+2:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[m:2\sigma_1,\ldots,2\sigma_n],\left[\frac{m-2}{2}:\sigma_1,\ldots,\sigma_n\right]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [m-p:2\sigma_1,\ldots,2\sigma_n],\left[\frac{m+p+1}{2}:\sigma_1,\ldots,\sigma_n\right]:[(d'):\delta'];\ldots;(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\ldots,z_n\right).$$

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CHAPTER-IX

MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST AND ITS APPLICATIONS IN TWO BOUNDARY VALUE PROBLEMS

9.1 INTRODUCTION.

Recently Chandel and Yadava [1] and Chandel and Gupta [2], have employed the hypergeometric function of Srivastava and Daoust [5, 6, 7,] (Also see for modified form Srivastava and Karlsson [8, p.37 eqs.(21) to (23)]) in different problems on heat conduction. In this chapter, first we evaluate a new integral involving the multiple hypergeometric function of Srivastava and Daoust [5, 6, 7] and its application will be made to derive solution of

(1) A problem on heat conduction in a rod.

(2) A problem on deflection of vibrating string under certain boundary conditions.

9.2 FORMULA REQUIRED.

In this chapter, we shall make an application of the following modified form of the integral [3, p.372 (1)]:

$$(9.2.1)\qquad \int_0^L \left(\operatorname{Sin}\frac{\pi x}{L}\right)^{\omega-1} \operatorname{Sin}\frac{\lambda_m \pi x}{L}\,dx = \frac{\omega L \operatorname{Sin}\pi\lambda_m/2}{2^{\omega-1}\,\Gamma\left(\frac{\omega+\lambda_m+1}{2}\right)\Gamma\left(\frac{\omega-\lambda_m+1}{2}\right)}\qquad R_e(\omega)>0$$

9.3 INTEGRAL

Making an appeal to (9.2.1), we obtain

$$(9.3.1)\qquad \int_0^L \left(\operatorname{Sin}\frac{\pi x}{L}\right)^{\omega-1} \operatorname{Sin}\frac{\lambda_m \pi x}{L}\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]: \\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1\left(\operatorname{Sin}\frac{\pi x}{L}\right)^{2\xi_1},\ldots,z_n\left(\operatorname{Sin}\frac{\pi x}{L}\right)^{2\xi_n}\right]dx$$

$$= \frac{L\operatorname{Sin}\frac{\pi\lambda_m}{2}}{2^{\omega-1}}\, S^{A+1:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$[\omega:2\xi_1,\ldots,2\xi_n]:$$

$$[\frac{\omega+1+\lambda_m}{2}:\xi_1,\ldots,\xi_n],[\frac{\omega+1-\lambda_m}{2}:\xi_1,\ldots,\xi_n]:$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1/4\xi_1,\ldots,z_n/4\xi_n\right],$$

provided that $R_e(\omega) > 0$,

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)} > 0,$$

and all ξ_i are real positive integers. This integer will be used in our further investigations.

PROBLEM - 1

9.4 APPLICATION TO HEAT CONDUCTION IN A ROD.

In this section, we consider a problem on outer heat conduction in a rod under certain boundary conditions. If the thermal coefficients are constants and there is no source of thermal energy, then the temperature u(x, t) in a one dimensional rod $0 \le x \le L$ satisfies the following heat equation:

$$(9.4.1) \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad t \ge 0.$$

If we taken the following boundary conditions

$$(9.4.2) \qquad u(0, t) = 0,$$

$$(9.4.3) \qquad \frac{\partial u}{\partial x}(L, t) + hu(L, t) = 0,$$

(9.4.4) $\qquad u(x, t)$ is finite as $t \to \infty$,

and initial condition

$$(9.4.5) \qquad u(x, 0) = f(x),$$

then the solution of partial differential equation (9.4.1) is given by [4, p.77, (4)]

$$(9.4.6) \qquad u(x, t) = \sum_{m=1}^{\infty} B_m \operatorname{Sin}\frac{\lambda_m \pi x}{L} \exp\left\{-\left(\frac{\pi\lambda_m}{L}\right)^2 kt\right\},$$

where $\lambda_1, ..., \lambda_m$ are the roots of the transcendental equation

$$(9.4.7) \qquad \tan\pi\lambda_m = \frac{\pi\lambda_m}{kL}.$$

Now we shall consider the problem of determining u(x, t), where

$$(9.4.8) \qquad u(x, 0) = f(x)$$

$$= \left(\operatorname{Sin}\frac{\pi x}{L}\right)^{\omega-1} S^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}]: \\ [(c):\psi',...,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];...;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1\left(\operatorname{Sin}\frac{\pi x}{L}\right)^{2\xi_1}, ..., z_n\left(\operatorname{Sin}\frac{\pi x}{L}\right)^{2\xi_n}\right].$$

9.5 SOLUTION OF THE PROBLEM.

Combining (9.4.6) and (9.4.8) and making the use of integral (9.3.1), we derive

$$(9.5.1) \qquad B_m = \frac{\pi\lambda_m \operatorname{Sin}\frac{\pi\lambda_m}{2}}{2^{\omega-3}\,[2\pi\lambda_m - \operatorname{Sin}2\pi\lambda_m]}$$

$$S^{A+1:B';...;B^{(n)}}_{C+2:D';...;D^{(n)}}\left(\begin{matrix}[(a):\theta',...,\theta^{(n)}],\,[\omega:2\xi_1,...,2\xi_n]; \\ [(c):\psi',...,\psi^{(n)}],\,[\frac{\omega+1+\lambda_m}{2}:\xi_1,...,\xi_n];\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];...;[(b^{(n)}):\Phi^{(n)}]; \\ [\frac{\omega+1-\lambda_m}{2}:\xi_1,...,\xi_n]:[(d'):\delta'];...;[(d^{(n)}):\delta^{(n)}]\end{matrix}\; Z_1/4^{\xi_1}, ..., Z_n/4^{\xi_n}\right],$$

where all the conditions of (9.3.1) are satisfied.

Putting the value of B_m from (9.5.1) in (9.4.6), we get the following required solution of the problem

(9.5.2)
$$u\ (x,t) = \frac{\pi}{2^{\omega-3}} \sum_{m=1}^{\infty} \text{Sin}\,\frac{\lambda_m \pi x}{L}\ \exp\left\{-\left(\frac{\pi \lambda_m}{2}\right)^2 kt\right\}$$
$$\frac{\lambda_m \,\text{Sin}\,\frac{\pi\lambda_m}{2}}{[2\pi\lambda_m - \text{Sin}\,2\pi\lambda_m]}\ S^{A+1:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$
$$[\omega : 2\xi_1,\ldots,2\xi_n]:$$
$$[\frac{\omega+1+\lambda_m}{2}:\xi_1,\ldots,\xi_n],[\frac{\omega+1-\lambda_m}{2}:\xi_1,\ldots,\xi_n]:$$
$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1/4\xi_1,\ldots,z_n/4\xi_n\right],$$

where all the conditions of (9.3.1) hold true.

9.6 EXPANSION FORMULA.

Making an use of (9.4.8) and (9.5.1) in (9.4.6) we derive the following expansion formula :

(9.6.1)
$$\left(\text{Sin}\,\frac{\pi x}{L}\right)^{\omega-1} S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}];\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1\left(\text{Sin}\,\frac{\pi x}{L}\right)^{2\xi_1},\ldots,z_n\left(\text{Sin}\,\frac{\pi x}{L}\right)^{2\xi_n}\right]$$
$$= \frac{\pi}{2^{\omega-3}} \sum_{m=1}^{\infty} \frac{\lambda_m\,\text{Sin}\,\frac{\pi\lambda_m}{2}\ \text{Sin}\,\frac{\lambda_m \pi x}{L}}{(2\pi\lambda_m - \text{Sin}\,2\pi\lambda_m)}\ S^{A+1:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$
$$[\omega : 2\xi_1,\ldots,2\xi_n]:$$
$$[\frac{\omega+1+\lambda_m}{2}:\xi_1,\ldots,\xi_n],[\frac{\omega+1-\lambda_m}{2}:\xi_1,\ldots,\xi_n]:$$
$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1/4\xi_1,\ldots,z_n/4\xi_n\right],$$

provided that all the conditions of (9.3.1) are satisfied.

PROBLEM - 2

9.7 APPLICATION TO HOMOGENEOUS WAVE PROBLEM.

In this section, we shall determine the shape (deflection) u(x, t) of vibrating string. If the deflection due to the weight of string in negligible (usually the case), then u(x, t) satisfies the partial deifferential equation

(9.7.1)
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L,\ t > 0.$$

Now we assume the boundary conditions

(9.7.2)
$$u(0,t) = 0,\quad u(L,t) = 0,\quad t > 0,$$

and initial conditions

(9.7.3)
$$\frac{\partial u}{\partial t}(x,0) = g(x)\ \ [\text{initial velocity}]$$

and

(9.7.4) $\quad u(x, 0) = f(x).$

Then the solution of partial differential equation (9.7.1) is given by

$$(9.7.5) \qquad u\ (x, t) = \sum_{m=1}^{\infty} \left(a_m \operatorname{Cos} \frac{\pi \lambda_m c t}{L} + b_m \operatorname{Sin} \frac{\pi \lambda_m c t}{L} \right) \operatorname{Sin} \frac{\pi \lambda_m x}{L},$$

Now we consider the problem of determininig u(x, t), where u(x, 0) = f(x) is given by (9.4.8) while

$$(9.7.6) \qquad g(x) = \left(\operatorname{Sin} \frac{\pi x}{L}\right)^{\omega' - 1} S^{E : F' ;...; F^{(n)}}_{G : H' ;...; H^{(n)}} \left(\begin{matrix} [(e) : \Theta', ..., \Theta^{(n)}] : \\ [(g) : \gamma', ..., \gamma^{(n)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(f') : Q'] ;...; [(f^{(n)}) : Q^{(n)}] ; \\ [(h') : \Omega'] ;...; [(h^{(n)}) : \Omega^{(n)}] ; \end{matrix} \; Z_1 \operatorname{Sin}^{2\rho_1} \frac{\pi x}{L}, ..., Z_n \operatorname{Sin}^{2\rho_n} \frac{\pi x}{L} \right]$$

By (9.7.3), (9.7.4) and (9.7.5), it is clear that

$$(9.7.7) \qquad u(x, 0) = f(x) = \sum_{m=1}^{\infty} a_m \operatorname{Sin} \frac{\pi \lambda_m x}{L}$$

and

$$(9.7.8) \qquad \frac{\partial u}{\partial t} (x, 0) = g(x) = \frac{\pi c}{L} \sum_{m=1}^{\infty} b_m \lambda_m \operatorname{Sin} \frac{\lambda_m x}{L}.$$

Now making an appeal to the integral (9.3.1), we find the values of a_m and b_m separately and put them in (9.7.5) to get required solution of the problem in the following form :

$$(9.7.9) \qquad u(x, t) = \frac{\pi}{2^{\omega - 3}} \sum_{m=1}^{\infty} \frac{\operatorname{Cos} \frac{\pi \lambda_m c t}{L} \operatorname{Sin} \frac{\pi \lambda_m x}{L} \operatorname{Sin} \frac{\pi \lambda_m}{2}}{(2 \pi \lambda_m - \operatorname{Sin} 2 \pi \lambda_m)}$$

$$S^{A+1 : B' ;...; B^{(n)}}_{C+2 : D' ;...; D^{(n)}} \left(\begin{matrix} [(a) : \Theta', ..., \Theta^{(n)}], [\omega : 2\xi_1, ..., 2\xi_n] ; \\ [(c) : \psi', ..., \psi^{(n)}], [\frac{\omega + 1 + \lambda_m}{2} : \xi_1, ..., \xi_n] ; \end{matrix} \right.$$

$$[(b') : \Phi'] ;...; [(b^{(n)}) : \Phi^{(n)}] ;$$

$$\left. [\frac{\omega + 1 - \lambda_m}{2} : \xi_1, ..., \xi_n] : [(d') : \delta'] ;...; [(d^{(n)}) : \delta^{(n)}] \quad Z_1/4\xi_1, ..., Z_n/4\xi_n \right]$$

$$+ \frac{L}{2^{\omega' - 3} . c} \sum_{m=1}^{\infty} \frac{\operatorname{Sin} \frac{\pi \lambda_m}{2} \operatorname{Sin} \frac{\pi \lambda_m c t}{L} \operatorname{Sin} \frac{\pi \lambda_m x}{L}}{[2 \pi \lambda_m - \operatorname{Sin} 2 \pi \lambda_m]}$$

$$S^{E+1 : F' ;...; F^{(n)}}_{G+2 : H' ;...; H^{(n)}} \left(\begin{matrix} [(e) : \Theta', ..., \Theta^{(n)}] , \\ [(g) : \gamma', ..., \gamma^{(n)}] , \end{matrix} \right.$$

$$[\omega' : 2\rho_1, ..., 2\rho_n] :$$

$$[\frac{\omega' + 1 + \lambda_m}{2} : \rho_1, ..., \rho_n], [\frac{\omega + 1 - \lambda_m}{2} : \rho_1, ..., \rho_n] :$$

$$\left. \begin{matrix} [(f') : Q'] ;...; [(f^{(n)}) : Q^{(n)}] ; \\ [(h') : \Omega'] ;...; [(h^{(n)}) : \Omega^{(n)}] ; \end{matrix} \; Z_1/4\rho_1, ..., Z_n/4\rho_n \right]$$

where All $R_e(\omega) > 0$,

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}>0,$$

$$1+\sum_{j=1}^{G}\gamma_j^{(i)}+\sum_{j=1}^{H^{(i)}}\Omega_j^{(i)}-\sum_{j=1}^{E}\Theta_j^{(i)}-\sum_{j=1}^{F^{(i)}}Q_j^{(i)}>0,$$

and all ξ_i, ρ_i are real positive integers, $i = 1,\ldots, n$.

9.8 SPECIAL CASE For initial velocity

(9.8.1) $$\frac{\partial u}{\partial t}(x,0)=g(x)=0.$$

all b's in (9.7.5) will be zero. Thus our problem 2 now reduces to the problem 1. Therefore, making an appeal to the integral (9.3.1), solution of the problem is given by

(9.8.2) $$u(x,t)=\frac{\pi}{2^{\omega-3}}\sum_{m=1}^{\infty}\frac{\lambda_m \operatorname{Cos}\frac{\pi\lambda_m ct}{L}\operatorname{Sin}\frac{\pi\lambda_m x}{L}\operatorname{Sin}\frac{\pi\lambda_m}{2}}{(2\pi\lambda_m-\operatorname{Sin}2\pi\lambda_m)}$$

$$S^{A+1:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\Theta',\ldots,\Theta^{(n)}],[\omega:2\xi_1,\ldots,2\xi_n],\\ [(c):\psi',\ldots,\psi^{(n)}],[\frac{\omega+1+\lambda_m}{2}:\xi_1,\ldots,\xi_n],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [\frac{\omega+1-\lambda_m}{2}:\xi_1,\ldots,\xi_n]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\;Z_1/4\xi_1,\ldots,Z_n/4\xi_n\right].$$

where all conditions of (9.3.1) are satisfied.

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CHAPTER - X

APPLICATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST AND THE MULTIVARIABLE H-FUNCTION OF SRIVASTAVA AND PANDA IN SOLVING A POTENTIAL PROBLEM ON A CIRCULAR DISK

In the present chapter, we first evaluate an integral involving hypergeometric function of Srivastava and Daoust ([7], [8]) and the multivariable H-function of Srivastava and Panda ([11], [12], [13]); and then we make an application of this integral to derive the solution of a potential problem on circular disk. Finally, we derive an expansion formula involving the product of multiple hypergeometric function of Srivastava and Daoust and the multivariable H-function of Srivastava and Panda.

10.1 INTRODUCTION. Recently, Mishra [5] evaluated an integral involving exponential function, sine function, two generalized hypergeometric series and Fox'sH-function [3]. Further Chandel, Agrawal and Pal [1] extended the work and evaluated an integral involving sine function, exponential function, Kampé de Fériet function [4] (also see Srivastava and Karlsson [9, p.27, (28)]) and the multivariable H-function of Srivastava and Panda ([11], [12], [13]). (For the integral also see Chandel, Agrawal and Kumar [2,(2.1)]). They applied this integral to derive the solution of a potential problem involving multivariable H- function of Srivastava and Panda ([11], [12], [13]). They also derived an expansion formula involving the product of the Kampé de Fériet function amd the mutlivariable H-function of Srivatava and Panda.

In the present chapter, we further extend the work and evaluate an interesting integral involving the sine function, exponential function, multiple hypergeometric function of Srivastava and Daoust ([7], [8]) (Also see Srivastava and Manocha [10, p.64 (18), (19)]), and multivariable H-function of Srivastava and Panda ([11], [12], [13]). We then apply this integral to derive the solution of a potential problem on a circular disk involving multivariable H-function of Srivastava and Panda. Also, we finally derive an expansion formula involving the product of above multiple hypergeometric function of Srivastava and Daoust and multivariable H-function of several variables of Srivastava and Panda. It is remarkable that all the results of Chandel, Agarwal and Pal [1], are special cases of the results of the present chapter.

10.2. MAIN INTEGRAL. In this section, we evaluate the following interesting integral:

$$(10.2.1)\quad \int_0^{\pi} \sin^{\omega-1}\theta\, e^{im\theta}\, F^{E:F';\dots;F^{(r)}}_{G:H';\dots;H^{(r)}}\left[\begin{matrix}[(e):\xi';\dots;\xi^{(r)}]:[(f'):\eta'];\\ [(g)]:\zeta';\dots;\zeta^{(r)}]:[(h'):\varepsilon'];\end{matrix}\right.$$

$$\left.\begin{matrix}\dots;[(f^{(r)}):\eta^{(r)}];\\ \dots;[(h^{(r)}):\varepsilon^{(r)}];\end{matrix}\ a_1\sin^{2\rho_1}\theta,\dots,a_r\sin^{2\rho_r}\theta\right]$$

$$H^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1\sin^{2\sigma_1}\theta,\dots,z_n\sin^{2\sigma_n}\theta\right]d\theta$$

$$= \frac{\pi e^{i m \pi/2}}{2^{\omega-1}} \sum_{m_1,\dots,m_r=0}^{\infty} \frac{\prod_{j=1}^{E} (e_j, m_1 \xi_j' + \dots + m_r \xi_j^{(r)}) \prod_{j=1}^{F'} (F'_j, m_1 \eta'_j) \dots}{\prod_{j=1}^{G} (g_j, m_1 \zeta'_j + \dots + m_r \zeta_j^{(r)}) \prod_{j=1}^{H'} (h'_j, m_1 \varepsilon'_j) \dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}} (F_j^{(r)}, m_r \eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}} (h_j^{(r)}, m_r \varepsilon_j^{(r)})} \frac{(a_1/4^{\rho_1})^{m_1}}{m_1!} \dots \frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$H^{0, \lambda+1 : (\mu', \nu') ;\dots; (\mu^{(n)}, \nu^{(n)})}_{A+1, C+2 : [B', D'] ;\dots; [B^{(n)}, D^{(n)}]} \left[\begin{matrix} [(a) : \theta', \dots, \theta^{(n)}], \\ [(c) : \psi', \dots, \psi^{(n)}], \end{matrix} \right.$$

$$[1 - \omega - 2(\rho_1 m_1 + \dots + \rho_r m_r) : 2\sigma_1, \dots, 2\sigma_n] : [(b') : \Phi'] ;\dots;$$

$$[\frac{1 - \omega - 2(\rho_1 m_1 + \dots + \rho_r m_r) \pm m}{2} : \sigma_1, \dots, \sigma_n] : [(d') : \delta'] ;\dots;$$

$$\left. \begin{matrix} [(b^{(n)}) : \phi^{(n)}]; \\ [(d^{(n)}) : \delta^{(n)}]; \end{matrix} \frac{z_1}{4^{\sigma_1}}, \dots, \frac{z_n}{4^{\sigma_n}} \right] ,$$

provided that $\arg(z_i) < \frac{\pi}{2} \Delta_i$, where (see [14, p.252])

$$\Delta_i = - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,$$

$i = 1, \dots, n.$

and

$$1 + \sum_{j=1}^{G} \zeta_j^{(k)} + \sum_{j=1}^{H^{(k)}} \varepsilon_j^{(k)} - \prod_{j=1}^{E} \xi_j^{(k)} - \sum_{j=1}^{F^{(k)}} \eta_j^{(k)} > 0,$$

$k = 1, \dots, r,$

$Re(\omega) > 0$, while $\sigma_1, \dots, \sigma_n ; \rho_1, \dots, \rho_r$ are positive real numbers and $a_1, \dots, a_r ; z_1, \dots, z_n$ and m are any real numbers. Here $F^{E : F' ;\dots; F^{(r)}}_{G : H' ;\dots; H^{(r)}}$ stands for multiple hypergeometric function of Srivastava and Daoust ([7], [8]).

$H^{0, \lambda : (\mu', \nu') ;\dots; (\mu^{(n)}, \nu^{(n)})}_{A, C : [B', D'] ;\dots; [B^{(n)}, D^{(n)}]}$ stands for multivariable H-function of Srivastava and Panda ([11], [12], [13]), while $[a \pm b ; \sigma_1, \dots, \sigma_n]$ abbreviates $[a + b : \sigma_1, \dots, \sigma_n]$, $[a - b : \sigma_1, \dots, \sigma_n]$.

Our result (10.2.1) includes the main result due to Chandel, Agrawal and Pal ([1, (3.1)], [2, (2.1)]) as special case for r = 2,

$$\xi_j^1 = \xi_j^{(2)} = 1 \quad (j = 1,\dots, E)\,;\ \zeta_j^1 = \zeta_j^{(2)} = 1 \ (j = 1,\dots, G)\,;$$

$$\varepsilon_j^1 = \varepsilon_j^{(2)} = 1 \ (j = 1,\dots, H')\,;\ \eta_j^1 = \eta_j^{(2)} = 1 \ (j = ,\dots, F')\,;.$$

Thus our result (10.2.1) also includes other results due to Chandel, Agrawal and Pal [1, (4.1), (4.2), (4.3), (4.4)] as special cases.

10.3 SPECIAL CASES. In this section, we give those special cases of (10.2.1), which will be useful in our further investigation. Equating the real and imaginary parts both the sides of (10.2.1), we derive

(10.3.1)
$$\int_0^{\pi} \sin^{\omega-1}\theta \cos m\theta \; F^{E:F';\dots;F^{(r)}}_{G:H';\dots;H^{(r)}}\left[\begin{matrix}[(e):\xi';\dots;\xi^{(r)}]:[(f'):\eta'];\dots;[(f^{(r)}):\eta^{(r)}];\\ [(g)]:\zeta';\dots;\zeta^{(r)}]:[(h'):\varepsilon'];\dots;[(h^{(r)}):\varepsilon^{(r)}];\end{matrix}\; a_1\sin^{2\rho_1}\theta,\dots,a_r\sin^{2\rho_r}\theta\right]$$

$$H^{0,\lambda:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1\sin^{2\sigma_1}\theta,\dots,z_n\sin^{2\sigma_n}\theta\right]d\theta$$

$$= \frac{\pi\cos\frac{m\pi}{2}}{2^{\omega-1}} \sum_{m_1,\dots,m_r=0}^{\infty} \frac{\prod_{j=1}^{E}(e_j, m_1\xi_j'+\dots+m_r\xi_j^{(r)}) \prod_{j=1}^{F'}(F'_j, m_1\eta'_j)\dots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j+\dots+m_r\zeta_j^{(r)}) \prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})} \; \frac{(a_1/4^{\rho_1})^{m_1}}{m_1!}\dots\frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$H^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-\omega-2(\rho_1m_1+\dots+\rho_rm_r):2\sigma_1,\dots,2\sigma_n]:[(b'):\Phi'];\dots;$$

$$\left[\frac{1-\omega-2(\rho_1m_1+\dots+\rho_rm_r)\pm m}{2}:\sigma_1,\dots,\sigma_n\right]:[(d'):\delta'];\dots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\;\frac{z_1}{4^{\sigma_1}},\dots,\frac{z_n}{4^{\sigma_n}}\right],$$

provided that all the conditions of (10.2.1) are satisfied.

$$(10.3.2) \quad \int_0^{\pi} \sin^{\omega-1}\theta \, \sin m\theta \; F^{E:F';...;F^{(r)}}_{G:H';...;H^{(r)}} \left[\begin{matrix} [(e):\xi';...;\xi^{(r)}]:[(f'):\eta'];\\ [(g)]:\zeta';...;\zeta^{(r)}]:[(h'):\varepsilon'\;]; \end{matrix} \right.$$

$$\left. \begin{matrix} ....;[(f^{(r)}):\eta^{(r)}];\\ ...;[(h^{(r)}):\varepsilon^{(r)}]; \end{matrix} \; a_1 \sin^{2\rho_1}\theta ,..., a_r \sin^{2\rho_r}\theta \right]$$

$$H^{0,\lambda:(\mu',\nu');...;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];...;[B^{(n)},D^{(n)}]} \left[\begin{matrix} [(a):\theta',...,\theta^{(n)}]:[(b'):\phi'];...;\\ [(c):\psi',...,\psi^{(n)}]:[(d'):\delta'];...; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}]; \end{matrix} \; z_1 \sin^{2\sigma_1}\theta ,..., z_n \sin^{2\sigma_n}\theta \right] d\theta$$

$$= \frac{\pi \sin \frac{m\pi}{2}}{2^{\omega-1}} \sum_{m_1,...,m_r=0}^{\infty} \frac{\prod_{j=1}^{E} (e_j, m_1\xi_j' +...+ m_r\xi_j^{(r)}) \prod_{j=1}^{F'} (f'_j, m_1\eta'_j) \cdots}{\prod_{j=1}^{G} (g_j, m_1\zeta'_j +...+ m_r\zeta_j^{(r)}) \prod_{j=1}^{H'} (h'_j, m_1\varepsilon'_j) \cdots}$$

$$\frac{\prod_{j=1}^{F^{(r)}} (f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}} (h_j^{(r)}, m_r\varepsilon_j^{(r)})} \quad \frac{(a_1/4^{\rho_1})^{m_1}}{m_1!} \cdots \frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$H^{0,\lambda+1:(\mu',\nu');...;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];...;[B^{(n)},D^{(n)}]} \left[\begin{matrix} [(a):\theta',...,\theta^{(n)}],\\ [(c):\psi',...,\psi^{(n)}], \end{matrix} \right.$$

$$[1-\omega-2(\rho_1 m_1 +...+ \rho_r m_r):2\sigma_1,...,2\sigma_n]:[(b'):\Phi'];...;$$

$$[\frac{1-\omega-2(\rho_1 m_1 +...+ \rho_r m_r) \pm m}{2}:\sigma_1,...,\sigma_n]:[(d'):\delta'];...;$$

$$\left. \begin{matrix} [(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}]; \end{matrix} \; \frac{z_1}{4^{\sigma_1}} ,..., \frac{z_n}{4^{\sigma_n}} \right],$$

valid if all the condition of (10.2.1) are satisfied.

For $m = 0$, (10.2.1) gives

$$(10.3.3) \quad \int_0^{\pi} \sin^{\omega-1}\theta \; F^{E:F';...;F^{(r)}}_{G:H';...;H^{(r)}} \left[\begin{matrix} [(e):\xi';...;\xi^{(r)}]:[(f'):\eta'];\\ [(g)]:\zeta';...;\zeta^{(r)}]:[(h'):\varepsilon']; \end{matrix} \right.$$

$$\left. \begin{matrix} ....;[(f^{(r)}):\eta^{(r)}];\\ ...;[(h^{(r)}):\varepsilon^{(r)}]; \end{matrix} \; a_1 \sin^{2\rho_1}\theta ,..., a_r \sin^{2\rho_r}\theta \right]$$

$$H^{0,\lambda:(\mu',\nu');...;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];...;[B^{(n)},D^{(n)}]} \left[\begin{matrix} [(a):\theta',...,\theta^{(n)}]:[(b'):\phi'];...;\\ [(c):\psi',...,\psi^{(n)}]:[(d'):\delta'];...; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}]; \end{matrix} \; z_1 \sin^{2\sigma_1}\theta ,..., z_n \sin^{2\sigma_n}\theta \right] d\theta$$

$$= \pi \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\prod_{j=1}^{E} (e_j, m_1 \xi'_j + \ldots + m_r \xi_j^{(r)}) \prod_{j=1}^{F'} (f'_j, m_1 \eta'_j) \ldots}{\prod_{j=1}^{G} (g_j, m_1 \zeta'_j + \ldots + m_r \zeta_j^{(r)}) \prod_{j=1}^{H'} (h'_j, m_1 \varepsilon'_j) \ldots}$$

$$\frac{\prod_{j=1}^{F^{(r)}} (F_j^{(r)}, m_r \eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}} (h_j^{(r)}, m_r \varepsilon_j^{(r)})} \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_r^{m_r}}{m_r!}$$

$$H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+1:[B',D'];\ldots;[B^{(n)},D^{(n)}]} \left[\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}], \\ [(c):\psi',\ldots,\psi^{(n)}], \end{matrix} \right.$$

$$[1-\frac{\omega}{2}-\rho_1 m_1-\ldots-\rho_r m_r : \sigma_1,\ldots,\sigma_n] : [(b'):\Phi'];\ldots;$$

$$[\frac{1-\omega}{2}-\rho_1 m_1-\ldots-\rho_r m_r : \sigma_1,\ldots,\sigma_n] : [(d'):\delta'];\ldots;$$

$$\left. \begin{matrix} [(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}]; \end{matrix} z_1,\ldots,z_n \right],$$

where all the conditions of (10.2.1) are satisfied.

10.4. A Potential Problem on a Circular Disk.

A problem involving potential that appears to be inhomogenous, but actually not, is the potential equation in circular disk specified all around the circumference. It is stated as follows:

(10.4.1) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 \le r \le L, \; -\pi \le \theta \le \pi$$

and

(10.4.2) $$u(L, \theta) = f(\theta), \qquad -\pi < \theta \le \pi$$

where

(10.4.3) $$\begin{aligned} f(\theta) &= 0, & \pi < \theta < 0 \\ &= u_0, & 0 < \theta < \pi \end{aligned}$$

There are following two peculiaries to this problem :

First is that the rays $\theta = \pi$ and $\theta = -\pi$ actually coincide. Thus the values of u and its angular derivatives should be same at $\theta = \pi$ and $\theta = -\pi$. Thus

(10.4.4) $$u(r, \pi) = u(r, -\pi) \text{ and } \frac{\partial u(r, -\pi)}{\partial \theta} = \frac{\partial u(r, \pi)}{\partial \theta}, \qquad 0 \le r \le L.$$

The second is that the point r = 0 is singular point; the coefficients of $\frac{\partial^2 u}{\partial r^2}$ in equation (10.4.1) is 1, while coefficients of other terms are $\frac{1}{r}$ and $\frac{1}{r^2}$, therefore we must except to enforce a boundedness condition

(10.4.5) $\quad u(r, \theta)$ is bounded as $r \to 0^+$

10.5. Solution of the problem. The solution of this problem given by Powers [6, p.611, (10.179)] is as follows:

$$(10.5.1) \qquad u(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m \,(a_m \cos m\,\theta + b_m \sin m\,\theta).$$

for r = L

$$(10.5.2) \qquad u(L,\theta) = f(\theta) = a_0 + \sum_{m=1}^{\infty} L^m \,(a_m \cos m\,\theta + b_m \sin m\,\theta).$$

For special interest, we shall find the solution of the problem when

$$(10.5.3) \qquad f(\theta) = \sin^{\omega-1}\theta\; g(\theta)\; \mathbf{F}^{E:F';\ldots;F^{(r)}}_{G:H';\ldots;H^{(r)}} \left[\begin{matrix} [(e):\xi';\ldots;\xi^{(r)}]:[(f'):\eta'];\ldots;[(f^{(r)}):\eta^{(r)}]; \\ [(g)]:\zeta';\ldots;\zeta^{(r)}]:[(h'):\varepsilon'];\ldots;[(h^{(r)}):\varepsilon^{(r)}]; \end{matrix}\; a_1 \sin^{2\rho_1}\theta,\ldots,a_r \sin^{2\rho_r}\theta \right]$$

$$\mathbf{H}^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]} \left[\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix}\; z_1 \sin^{2\sigma_1}\theta,\ldots,z_n \sin^{2\sigma_n}\theta \right],$$

where

$$(10.5.4) \qquad g(\theta) = 0 \;, \quad -\pi < \theta < 0$$
$$\qquad = U_0 \text{ (Constant)}, \; 0 < \theta\, \pi.$$

This is seen to be a Fourier series problem. Thus

$$(10.5.5) \qquad a_0 = \frac{1}{2\pi}\int_0^{\pi} f(\theta)\, d\,\theta$$

$$(10.5.6) \qquad a_m = \frac{1}{L^m \pi}\int_0^{\pi} f(\theta) \cos m\,\theta \;d\,\theta$$

and

$$(10.5.7) \qquad b_m = \frac{1}{L^m \pi}\int_0^{\pi} f(\theta) \sin m\,\theta \;d\,\theta.$$

Now substituting the values of f(θ) from (10.5.3) in (10.5.5), (10.5.6) and (10.5.7) and making an appeal to (10.3.3), (10.3.1) and (10.3.2) respectively, we derive

$$(10.5.8)\qquad a_0 = \frac{U_0}{2}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\prod_{j=1}^{E}(e_j, m_1\xi'_j+\dots+m_r\xi_j^{(r)})\prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\dots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j+\dots+m_r\zeta_j^{(r)})\prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})}\;\frac{a_1^{m_1}}{m_1!}\cdots\frac{a_r^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+1:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-\frac{\omega}{2}-\rho_1 m_1-\dots-\rho_r m_r:\sigma_1,\dots,\sigma_n]:[(b'):\Phi'];\dots;$$

$$[\frac{1-\omega}{2}-\rho_1 m_1-\dots-\rho_r m_r:\sigma_1,\dots,\sigma_n]:[(d'):\delta'];\dots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\dots,z_n\right],$$

where all the conditions of (10.2.1) are satisfied.

$$(10.5.9)\qquad a_m = \frac{U_0}{L^m}\,\frac{\cos\frac{m\pi}{2}}{2^{\omega-1}}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\prod_{j=1}^{E}(e_j, m_1\xi'_j+\dots+m_r\xi_j^{(r)})\prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\dots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j+\dots+m_r\zeta_j^{(r)})\prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(\mathcal{F}_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})}\;\frac{(a_1/4^{\rho_1})^{m_1}}{m_1!}\cdots\frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-\omega'-2(\rho_1 m_1+\dots+\rho_r m_r):2\sigma_1,\dots,2\sigma_n]:[(b'):\Phi'];\dots;$$

$$[\frac{1-\omega'-2(\rho_1 m_1+\dots+\rho_r m_r)\pm m}{2}:\sigma_1,\dots,\sigma_n]:[(d'):\delta'];\dots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\;\frac{z_1}{4^{\sigma_1}},\dots,\frac{z_n}{4^{\sigma_n}}\right],$$

provided that all the conditions of (10.2.1) are satisfied.

and

$$(10.5.10)\quad b_m = \frac{U_0}{L^m}\,\frac{\sin\frac{m\pi}{2}}{2^{\omega-1}}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\prod_{j=1}^{E}(e_j, m_1\xi'_j+\dots+m_r\xi_j^{(r)})\prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\dots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j+\dots+m_r\zeta_j^{(r)})\prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})}\;\frac{(a_1/4^{\rho_1})^{m_1}}{m_1!}\cdots\frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-\omega'-2(\rho_1 m_1+\dots+\rho_r m_r):2\sigma_1,\dots,2\sigma_n]:[(b'):\Phi'];\dots;$$

$$\left[\frac{1-\omega'-2(\rho_1 m_1+\dots+\rho_r m_r)\pm m}{2}:\sigma_1,\dots,\sigma_n\right]:[(d'):\delta'];\dots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\;\frac{z_1}{4^{\sigma_1}},\dots,\frac{z_n}{4^{\sigma_n}}\right],$$

valid if all the conditions of (10.2.1) are satisfied.

Therefore, substituting the value of a_0, a_m and b_m from (10.5.8), (10.5.9) and (10.5.10) respectively in (10.5.1), we obtain the following solution of the problem:

$$(10.5.11)\quad u(r,\theta) = \frac{U_0}{2}\sum_{m_1,\dots,m_r=0}^{\infty}\frac{\prod_{j=1}^{E}(e_j, m_1\xi'_j+\dots+m_r\xi_j^{(r)})\prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\dots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j+\dots+m_r\zeta_j^{(r)})\prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\dots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})}\;\frac{a_1^{m_1}}{m_1!}\cdots\frac{a_r^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+1:[B',D'];\dots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\psi',\dots,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-\frac{\omega}{2}-\rho_1 m_1-\dots-\rho_r m_r:\sigma_1,\dots,\sigma_n]:[(b'):\Phi'];\dots;\\ [\frac{1-\omega}{2}-\rho_1 m_1-\dots-\rho_r m_r:\sigma_1,\dots,\sigma_n]:[(d'):\delta'];\dots;\end{matrix}\;\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\;z_1,\dots,z_n\right]$$

$$+\frac{U_0}{2^{\omega-1}}\sum_{m=1}^{\infty}\;\sum_{m_1,\dots,m_r=0}^{\infty}\frac{r^m}{L^m}\cos m\left(\frac{\pi}{2}-\theta\right)$$

$$\frac{\prod_{j=1}^{E}(e_j, m_1\xi_j' + \ldots + m_r\xi_j^{(r)}) \prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\ldots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j + \ldots + m_r\zeta_j^{(r)}) \prod_{j=1}^{H'}(h'_j, m_1\varepsilon'_j)\ldots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})} \frac{(a_1/4^{\rho_1})^{m_1}}{m_1!}\cdots\frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-\omega'-2(\rho_1 m_1+\ldots+\rho_r m_r):2\sigma_1,\ldots,2\sigma_n]:[(b'):\Phi'];\ldots;$$

$$\left[\frac{1-\omega'-2(\rho_1 m_1+\ldots+\rho_r m_r)\pm m}{2}:\sigma_1,\ldots,\sigma_n\right]:[(d'):\delta'];\ldots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1}{4^{\sigma_1}},\ldots,\frac{z_n}{4^{\sigma_n}}\right],$$

where all the conditions of (10.2.1) are satisfied.

10.6. EXPANSION FORMULA. Making an appeal to (10.5.3), (10.5.8), (10.5.9) and (10.5.10), we derive the following expansion formula:

$$\sin^{\omega-1}\theta\, g(\theta)\, F^{E:F';\ldots;F^{(r)}}_{G:H';\ldots;H^{(r)}}\left[\begin{matrix}[(e):\xi';\ldots;\xi^{(r)}]:[(f'):\eta'];\\ [(g)]:\zeta';\ldots;\zeta^{(r)}]:[(h'):\varepsilon'];\end{matrix}\right. \tag{10.6.1}$$

$$\left.\begin{matrix}\ldots;[(f^{(r)}):\eta^{(r)}];\\ \ldots;[(h^{(r)}):\varepsilon^{(r)}];\end{matrix}\ a_1\sin^{2\rho_1}\theta,\ldots,a_r\sin^{2\rho_r}\theta\right]$$

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1\sin^{2\rho_1}\theta,\ldots,z_n\sin^{2\rho_n}\theta\right],$$

$$=\frac{U_0}{2}\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{\prod_{j=1}^{E}(e_j, m_1\xi_j' + \ldots + m_r\xi_j^{(r)}) \prod_{j=1}^{F'}(f'_j, m_1\eta'_j)\ldots}{\prod_{j=1}^{G}(g_j, m_1\zeta'_j + \ldots + m_r\zeta_j^{(r)}) \prod_{j=1}^{H}{}'(h'_j, m_1\varepsilon'_j)\ldots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)}, m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)}, m_r\varepsilon_j^{(r)})} \frac{a_1^{m_1}}{m_1!}\cdots\frac{a_r^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+1:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-\frac{\omega}{2}-\rho_1 m_1-\ldots-\rho_r m_r:\sigma_1,\ldots,\sigma_n]:[(b'):\Phi'];\ldots; & [(b^{(n)}):\phi^{(n)}]; \\ [\frac{1-\omega}{2}-\rho_1 m_1-\ldots-\rho_r m_r:\sigma_1,\ldots,\sigma_n]:[(d'):\delta'];\ldots; & [(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\ldots,z_n\right]$$

$$+\frac{U_0}{2^{\omega-1}}\sum_{m=1}^{\infty}\;\sum_{m_1,\ldots,m_r=0}^{\infty}\cos m\left(\frac{\pi}{2}-\theta\right)$$

$$\frac{\prod_{j=1}^{E}(e_j,m_1\xi_j'+\ldots+m_r\xi_j^{(r)})\prod_{j=1}^{F'}(f'_j,m_1\eta'_j)\ldots}{\prod_{j=1}^{G}(g_j,m_1\zeta'_j+\ldots+m_r\zeta_j^{(r)})\prod_{j=1}^{H'}(h'_j,m_1\varepsilon'_j)\ldots}$$

$$\frac{\prod_{j=1}^{F^{(r)}}(f_j^{(r)},m_r\eta_j^{(r)})}{\prod_{j=1}^{H^{(r)}}(h_j^{(r)},m_r\varepsilon_j^{(r)})}\;\frac{(a_1/4^{\rho_1})^{m_1}}{m_1!}\cdots\frac{(a_r/4^{\rho_r})^{m_r}}{m_r!}$$

$$\mathbf{H}^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+2:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],\\ [(c):\psi',\ldots,\psi^{(r)}],\end{matrix}\right.$$

$$[1-\omega'-2(\rho_1 m_1+\ldots+\rho_r m_r):2\sigma_1,\ldots,2\sigma_n]:[(b'):\Phi'];\ldots;$$

$$[\frac{1-\omega'-2(\rho_1 m_1+\ldots+\rho_r m_r)\pm m}{2}:\sigma_1,\ldots,\sigma_n]:[(d'):\delta'];\ldots;$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\;\frac{z_1}{4^{\sigma_1}},\ldots,\frac{z_n}{4^{\sigma_n}}\right],$$

valid if all the conditions of (10.2.1) are satisfied

and

$$\begin{aligned} g(\theta) &= 0, \quad -\pi<\theta<0 \\ &= U_0, \quad 0<\theta<\pi. \end{aligned}$$

Finally, we remark that for $r = 2$, $\xi_j^{!} = \xi_j^{(2)} = 1$

$(j = 1,\ldots, E)$; $= \zeta_j^{1} = \zeta_j^{(2)} = 1$ $(j = 1,\ldots, G)$; $\varepsilon_j^{1} = \varepsilon_j^{(2)} = 1$

$(j = 1,\ldots, H')$; $\eta_j^{1} = \eta_j^{(2)} = 1$ $(j = 1,\ldots, F')$,

the results of this chapter include all the results due to Chandel, Agrawal and Pal [1].

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Indian J. pure appl. Math., 22(9): 757-761, September 1991

MULTIVARIABLE ANALOGUE OF GOULD AND HOPPER'S POLYNOMIALS DEFINED BY RODRIGUES' FORMULA

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(*Received* 19 *June* 1990; *after revision* 24 *April* 1991; *accepted* 1 *July* 1991)

In the present paper, we introduce a multivariable analogue of Gould and Hopper's polynomials[5] through Rodrigues' formula (1.1).

1. INTRODUCTION

Recently Chandel and Sahgal[2,3] have studied multivariable analogues of Panda's polynomials, and Gould's and Gould-Hopper's polynomials through their 'generating functions'. Motivated by the above works, in the present paper, we introduce and study the multivariable analogue

$$G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

of Gould and Hopper's polynomials[5] through Rodrigues' formula

$$G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= (-1)^{n_1+\ldots+n_m} x_1^{-a_1} \ldots x_m^{-a_m} [G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})]^{-1}$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \ldots \frac{d^{n_m}}{dx_m^{n_m}} \{x_1^{a_1} \ldots x_m^{a_m} G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})\} \qquad \ldots(1.1)$$

where parameters $r_1,\ldots,r_m;\ a_1,\ldots,a_m,\ p_1,\ldots,\ p_m$ are unrestricted in genera but independent of variables $x_1,\ldots,\ x_m$ and

$$G(z) = \sum_{n=0}^{\infty} \gamma_n z^n,\ \gamma_0 \neq 0. \qquad \ldots(1.2)$$

2. GENERATING FUNCTION

Replacing each x_i by $1/x_i$, $i = 1,\ldots,m$, we derive from (1.1)

$$\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(1/x_1,\ldots,1/x_m)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= x_1^{a_1}\ldots\ldots x_m^{a_m}\,[G\,(p_1\,x_1^{-r_1}+\ldots+p_m\,x_m^{-r_m})]^{-1}$$

$$\exp\left(t_1\,\Omega_{x_1}+\ldots+t_m\,\Omega_{x_m}\right)\left\{x_1^{-a_1}\ldots x_m^{-a_m}\,G(p_1\,x_1^{-r_1}+\ldots+p_m\,x_m^{-r_m})\right\} \qquad \ldots(2.1)$$

which by making an appeal to the result due to Chandel and Agarwal[1] [p. 88(3.2)] (Also see earlier reference due to Edwards[4] [p. 506 Misc. Ex. No. 15])

$$e^{t\Omega_x}\{f(x)\} = f\left(\frac{x}{1-xt}\right),\ \Omega_x = x^2\,\frac{\partial}{\partial x}$$

finally gives the generating relation

$$\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)\,\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left(1-\frac{t_1}{x_1}\right)^{a_1}\cdots\left(1-\frac{t_m}{x_m}\right)^{a_m}\frac{G[p_1\,(x_1-t_1)^{r_1}+\ldots+p_m(x_m-t_m)^{r_m}]}{G(p_1\,x_1^{r_1}+\ldots+p_m\,x_m^{r_m})}. \qquad \ldots(2.2)$$

3. Explicit Form

Starting with the generating relation (2.2), we derive the following explicit form:

$$G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$=\frac{(-a_1)_{n_1}\cdots(-a_m)_{n_m}}{G(p_1\,x_1^{r_1}+\ldots+p_m\,x_m^{r_m})}\,\frac{1}{x_1^{n_1}\ldots x_m^{n_m}}\sum_{k_1,\ldots,k_m=0}^{\infty}\gamma_{k_1+\ldots+k_m}$$

$$\frac{(1+a_1)_{r_1k_1}\cdots(1+a_m)_{r_mk_m}}{(1+a_1-n_1)_{r_1k_1}\cdots(1+a_m-n_m)_{r_mk_m}}\,\frac{(p_1x_1^{r_1})^{k_1}}{k_1!}\cdots\frac{(p_mx_m^{r_m})^{k_m}}{k_m!}. \qquad \ldots(3.1)$$

4. Applications of Generating Relation

An appeal to generating relation (2.2) gives

$$G_{n_1,\ldots,n_m}^{(a_1+b_1,\ldots,a_m+b_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= \sum_{k_1=0}^{\min(n_1,[b_1])} \cdots \sum_{k_m=0}^{\min(n_m,[b_m])} \frac{(-b_1)_{k_1} \cdots (-b_m)_{k_m}}{x_1^{k_1} \cdots x_m^{k_m}} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m}$$

$$G_{n_1-k_1,\ldots,\,n_m-k_m}^{(a_1,\ldots,\,a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m). \qquad \ldots(4.1)$$

Again from generating relation (2.2), we derive the differential recurrence relation

$$\left(\frac{x_1^2 \frac{\partial}{\partial x_1}\left[G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})\right]}{G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})} + x_1^2 \frac{\partial}{\partial x_1}\right)$$

$$G_{n_1,\ldots,n_m}^{(a_1,\ldots,\,a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= n_1\, a_1\, G_{n_1-1,\, n_2,\ldots,n_m}^{(a_1-1,\, a_2,\ldots,\, a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$- a_1\, x_1\, G_{n_1,\ldots,n_m}^{(a_1-1,\, a_2,\ldots,\, a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$- x_1^2\, G_{n_1+1,\, n_2,\ldots,n_m}^{(a_1,\ldots,\, a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m). \qquad \ldots(4.2)$$

which suggests the m-results similar to above can be unified in the form

$$\left(\frac{x_i^2 \frac{\partial}{\partial x_i} G\,(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})}{G(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})} + x_i^2 \frac{\partial}{\partial x_i}\right)$$

$$G_{n_1,\ldots,n_m}^{(a_1,\ldots,\,a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= n_i\, a_i\, G_{n_1,\ldots,n_{i-1},\, n_i-1,\, n_{i+1},\ldots,n_m}^{(a_1,\ldots,\, a_{i-1},\, a_i-1,\, a_{i+1},\ldots,a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$- a_i\, x_i\, G_{n_1,\ldots,n_m}^{(a_1,\ldots,a_{i-1},\, a_i-1,\, a_{i+1},\ldots,\, a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$- x_i^2\, G_{n_1,\ldots,\, n_{i-1},\, n_i+1,\, n_{i+1},\ldots,n_m}^{(a_1,\ldots,\, a_m;\; r_1,\ldots,r_m;\; p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$i = 1,\ldots,\, m. \qquad \ldots(4.3)$$

5. Special Cases

Particularly for $\gamma_n = \dfrac{(-1)^n (b)_n}{n!}$, (1.1) defines

$$H_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m;\, b)}(x_1,\ldots,x_m)$$

$$= (-1)^{n_1+\ldots+n_m} x_1^{-a_1} \ldots x_m^{-a_m} [1+p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}]^b$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \ldots \frac{d^{n_m}}{dx_m^{n_m}} \left\{ x_1^{a_1} \ldots x_m^{a_m} (1+p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})^{-b} \right\} \qquad \ldots(5.1)$$

where parameters $r_1,\ldots,r_m$, $a_1,\ldots,a_m$, $p_1,\ldots,p_m$, b are unrestricted in general but independent of variables $x_1,\ldots,x_m$.

For $\gamma_n = (-1)^n/n!$, (1.1) defines

$$E_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= (-1)^{n_1+\ldots+n_m} x_1^{-a_1} \ldots x_m^{-a_m} \exp\{-(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})\}$$

$$\frac{d^{n_1}}{dx_1^{n_1}} \ldots \frac{d^{n_m}}{dx_m^{n_m}} \{x_1^{a_1} \ldots x_m^{a_m} \exp(p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m})\} \qquad \ldots(5.2)$$

It is clear that

$$\lim_{b\to\infty} H_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1/b,\ldots,p_m/b;\, b)}(x_1,\ldots,x_m)$$

$$= H_{n_1}^{r_1}(x_1, a_1, p_1) \ldots H_{n_m}^{r_m}(x_m, a_m, p_m) \qquad \ldots(5.3)$$

and

$$E_{n_1,\ldots,n_m}^{(a_1,\ldots,a_m;\, r_1,\ldots,r_m;\, p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= H_{n_1}^{r_1}(x_1, a_1, p_1) \ldots H_{n_m}^{r_m}(x_m, a_m, p_m) \qquad \ldots(5.4)$$

where $H_n^r(x, a, p)$ are Gould and Hopper's polynomials defined by Rodrigues' formula[5] [Also see Srivastava and Manocha[6], p. 77eq. (12)]

$$H_n^r(x, a, p) = (-1)^n x^{-a} e^{px^r} \frac{d^n}{dx^n} \{x^a e^{-px^r}\}. \qquad \ldots(5.5)$$

Acknowledgement

Authors are very much thankful to the referees for their valuable suggestions to bring the paper in the present form.

Ganita Sandesh Vol. 5, No. 2, 92–95 (1991)

A MULTIVARIABLE ANALOGUE OF HERMITE POLYNOMIALS

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1. INTRODUCTION

Recently, Beniwal and Saran [1] have studied two variable analogue $L_{m,n}^{(a,b,c)}(x, y)$ of Laguerre polynomials associated with Appell function F_2, and defined by

$$L_{m,n}^{(a,b,c)}(x, y, z) = \frac{(b)_m(c)_n}{m!\,n!} F_2[a, -m, -n; b, c; x, y], \tag{1.1}$$

from which it is clear that

$$\lim_{a\to\infty} L_{m,n}^{(a,b,c)}\left(\frac{x}{a}, \frac{y}{a}\right) = L_m^{(b-1)}(x)L_n^{(c-1)}(y) \tag{1.2}$$

Motivated by above work, very recently Raizada and Shrivastava [2] have defined two variable analogue $P_{k,n}^{(\nu)}(x, y)$ of Legendre polynomials by the integral

$$P_{k,n}^{(\nu)}(x, y) = \frac{2^2}{n!\,k!\,\pi}\int_0^\infty\int_0^\infty (\exp - (t^2 + T^2)t^kT^nH_{k,n}^{(\nu)}(xt, yt)\,dt\,dT. \tag{1.3}$$

where $H_{k,n}^{(\nu)}(x, y)$ is two variable analogue of Hermitė polynomials defined by Raizada and Shrivastava [3] in the following way:

$$\sum_{n,k=0}^{\infty} \frac{H_{k,n}^{(\nu)}(x, y)}{k!\,n!} t^kT^n = \exp[-(t^2 + T^2)](1 + 2xt + 2yT)^\nu \tag{1.4}$$

from (1.3) it is clear that

$$\lim_{\nu\to\infty} P_{k,n}^{(\nu)}\left(\frac{x}{\nu}, \frac{y}{\nu}\right) = p_k(x)\cdot p_n(y). \tag{1.5}$$

while from (1.4), it is clear that

$$\lim_{\nu\to\infty} H_{k,n}^{(\nu)}\left(\frac{x}{\nu}, \frac{y}{\nu}\right) = H_k(x)H_n(y), \tag{1.6}$$

where $P_n(x)$ and $H_n(x)$ are Legendre polynomials and Hermite polynomials respectively.

Motivated by the above works, in the present paper, we introduce the multivariable analogue $H_{n_1,\ldots,n_m}^{(b;\,h_1,\ldots,h_m;\,r_1,\ldots,r_m)}(x_1, \ldots, x_m)$ of Hermite polynomials, defined by Rodrigues's formula

$$\begin{aligned} &H_{n_1,\ldots,n_m}^{(b;\,h_1,\ldots,h_m;\,r_1,\ldots,r_m)}(x_1, \ldots, x_m) \\ &= (-1)^{n_1+\ldots+n_m}(1 + h_1x_1^{r_1} + \ldots + h_mx_m^{r_m})^b \\ &\qquad\times\frac{d^{n_1}}{dx_1^{n_1}}\cdots\frac{d^{n_m}}{dx_m^{n_m}}(1 + h_1x_1^{r_1} + \ldots + h_mx_m^{r_m})^{-b} \end{aligned} \tag{1.7}$$

where $n_1, \ldots, n_m$ are positive integers while $h_1, \ldots, h_m; r_1, \ldots, r_m$ and b are any numbers real or complex.

From (1.7) we have

$$\lim_{b\to\infty} H_{n_1\ldots, n_m}^{(b, 1/b, \ldots, 1/b; 2, \ldots, 2)}(x_1, \ldots, x_m) = H_{n_1}(x_1)\ldots, H_{n_m}(x_m), \tag{1.8}$$

where $H_n(x)$ are Hermite polynomials.

2. Generating Relation

Replacing x_i by $1/x_i$, $i = 1, \ldots, m$ in (1.7) and applying well known result

$$e^{tD_x}\{f(x)\} = f(x - t) \tag{2.1}$$

we derive generating relation

$$\sum_{n_1, \ldots, n_m=0}^{\infty} H_{n_1, \ldots, n_m}^{(b; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m) \frac{t_1^{n_1}}{n_1!} \ldots \frac{t_m^{n_m}}{n_m!}$$

$$= [1 + h_1x_1^{r_1} + \ldots + h_mx_m^{r_m}]^b[1 + h_1(x_1 - t_1)^{r_1} + \ldots + h_m(x_m - t_m)^{r_m}]^{-b}. \tag{2.2}$$

3. Application of Generating Relation

Making an appeal to generating relation (2.2), we obtain

$$H_{n_1, \ldots, n_m}^{(b+b'; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= \sum_{k_1=0}^{n_1} \ldots \sum_{k_m=0}^{n_m} \binom{n_1}{k_1} \ldots \binom{n_m}{k_m} H_{n_1-k_1, \ldots, n_m-k_m}^{(b; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$\times H_{k_1, \ldots, k_m}^{(b'; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots x_m). \tag{3.1}$$

Differentiating generating relation (2.2) w.r.t. t_1 and equating the coefficients of $t_1^{n_1} \ldots t_m^{n_m}$ on both the sides, we get

$$(1 + h_1x_1^{r_1} + \ldots + h_mx_m^{r_m})H_{n_1+1, n_2, \ldots, n_m}^{(b; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= bh_1r_1x^{r_1-1} \sum_{k=0}^{\min(r_1-1, n_1)} \binom{r_1 - 1}{k} \left(\frac{-1}{x_1}\right)^k \frac{n_1!}{(n_1 - k)!}$$

$$\times H_{n_1-k, n_2, \ldots, n_m}^{(b+1; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.2}$$

which can be further generalised in the form:

$$1 + h_1x_1^{r_1} + \ldots + h_mx_m^{r_m})H_{n_1, \ldots, n_{i-1}, n_i+1, n_{i+1}, \ldots, n_m}^{(b; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= bh_ir_i(x_i)^{r_i-1} \sum_{k=0}^{\min(r_i-1, n_i)} \binom{r_i - 1}{k} \left(\frac{-1}{x_i}\right)^k \frac{n_i!}{(n_i - k)!}$$

$$\times H_{n_1, \ldots, n_{i-1}, n_i-k, n_{i+1}, \ldots n_m}^{(b+1; h_1, \ldots, h_m; r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.3}$$

where $i = 1, \ldots, m$.

Now differentiating generating relation (2.2) partially w.r.t. x_1 and equating efficients of $t_1^{n_1} \dots t_m^{n_m}$ on both the sides, we establish

$$\left[br_1h_1x_1^{r_1-1} - (1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m})\frac{\partial}{\partial x_1}\right] H_{n_1, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots$$

$$= br_1h_1x_i^{r_1-1} \sum_{k=0}^{\min(r_1-1,\, n_1)} \binom{r_1-1}{k}\left(\frac{-1}{x_1}\right)^k \frac{n_1!}{(n_1-k)!}$$

$$H_{n_1, \dots, n_m}^{(b+1;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x$$

which can be generalized in the following form:

$$\left[br_ih_i(x_i)^{r_i-1} - (1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m})\frac{\partial}{\partial x_i}\right] H_{n_1, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots$$

$$= br_ih_i(x_i)^{r_i-1} \sum_{k=0}^{\min(r_i-1,\, n_i)} \binom{r_i-1}{k}\left(\frac{-1}{x_i}\right)^k \frac{n_i!}{(n_i-k)!}$$

$$H_{n_1, \dots, n_{i-1}, n_i+1, n_{i+1}, \dots, n_m}^{(b+1;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x$$

where $i = 1, \dots, m$.

combining (3.3) and (3.5) we further derive

$$\left[br_ih_i(x_i)^{r_i-1} - (1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m})\frac{\partial}{\partial x_i}\right] H_{n_1, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1,$$

$$= (1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m}) H_{n_1, \dots, n_{i-1}, n_i+1, n_{i+1}, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x$$

where $i = 1, \dots, m$.

From (3.6)

$$\left(\frac{br_ih_ix_i^{r_i-1}}{1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m}} - \frac{\partial}{\partial x_i}\right) H_{n_1, \dots n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x_m)$$

$$= H_{n_1, \dots, n_{i-1}, n_i+1, \dots, n_{i+1}, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots,$$

For take $\dfrac{br_ih_ix_i^{r_i-1}}{1 + h_1x_1^{r_1} + \dots + h_mx_m^{r_m}} - \dfrac{\partial}{\partial x_i} = \mathcal{S}_i$

$$\mathcal{S}_i\{H_{n_1, \dots n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x_m)\}$$

$$= H_{n_1, \dots, n_{i-1}, n_i+1, n_{i+1}, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots,$$

$$\mathcal{S}_i^j\{H_{n_1, \dots, n_m)}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x_m)\}$$

$$= H_{n_1, \dots, n_{i-1}, n_i+j, n_{i+1}, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x$$

which gives

$$e^t\mathcal{S}_i\{H_{n_1, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x_m)\}$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} H_{n_1, \dots, n_{i-1}, n_i+j, n_{i+1}, \dots, n_m}^{(b;\, h_1, \dots, h_m;\, r_1, \dots, r_m)}(x_1, \dots, x$$

Now differentiating generating relation (2.2) partially w.r.t. x_1 and equating the coefficients of $t_1^{n_1} \ldots t_m^{n_m}$ on both the sides, we establish

$$\left[b r_1 h_1 x_1^{r_1-1} - (1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}) \frac{\partial}{\partial x_1}\right] H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= b r_1 h_1 x_i^{r_1-1} \sum_{k=0}^{\min(r_1-1,\, n_1)} \binom{r_1-1}{k} \left(\frac{-1}{x_1}\right)^k \frac{n_1!}{(n_1-k)!}$$

$$H_{n_1, \ldots, n_m}^{(b+1;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m), \tag{3.4}$$

which can be generalized in the following form:

$$\left[b r_i h_i (x_i)^{r_i-1} - (1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}) \frac{\partial}{\partial x_i}\right] H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots x_m)$$

$$= b r_i h_i (x_i)^{r_i-1} \sum_{k=0}^{\min(r_i-1,\, n_i)} \binom{r_i-1}{k} \left(\frac{-1}{x_i}\right)^k \frac{n_i!}{(n_i-k)!}$$

$$H_{n_1, \ldots, n_{i-1}, n_i+1, n_{i+1}, \ldots, n_m}^{(b+1;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.5}$$

where $i = 1, \ldots, m$.
combining (3.3) and (3.5) we further derive

$$\left[b r_i h_i (x_i)^{r_i-1} - (1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}) \frac{\partial}{\partial x_i}\right] H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= (1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}) H_{n_1, \ldots, n_{i-1}, n_i+1, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.6}$$

where $i = 1, \ldots, m$.
From (3.6)

$$\left(\frac{b r_i h_i x_i^{r_i-1}}{1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}} - \frac{\partial}{\partial x_i}\right) H_{n_1, \ldots n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)$$

$$= H_{n_1, \ldots, n_{i-1}, n_i+1, \ldots, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.7}$$

For take $\dfrac{b r_i h_i x_i^{r_i-1}}{1 + h_1 x_1^{r_1} + \ldots + h_m x_m^{r_m}} - \dfrac{\partial}{\partial x_i} = \mathcal{S}_i$

$$\mathcal{S}_i\{H_{n_1, \ldots n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)\}$$

$$= H_{n_1, \ldots, n_{i-1}, n_i+1, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.8}$$

$$\mathcal{S}_i^j\{H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)\}$$

$$= H_{n_1, \ldots, n_{i-1}, n_i+j, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \tag{3.9}$$

which gives

$$e^t \mathcal{S}_i\{H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)\}$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} H_{n_1, \ldots, n_{i-1}, n_i+j, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m), \tag{3.10}$$

where $i = 1, \ldots, m$.

Specially for $j = n_j$ in (3.8), we have

$$\mathcal{S}_i^{n_j}\{H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots r_m)}(x_1, \ldots, x_m)\} = H_{n_1, \ldots, n_{i-1}, n_i+n_j, n_{i+1}, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \quad (3.11)$$

Also

$$\prod_{i=1}^{m} \mathcal{S}_i^{k_i}\{H_{n_1, \ldots, n_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m)\} = H_{n_1+k_1, \ldots, n_m+k_m}^{(b;\, h_1, \ldots, h_m;\, r_1, \ldots, r_m)}(x_1, \ldots, x_m) \quad (3.12)$$

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Pure And Applied Mathematika Sciences
Vol. XXXIV, No. 1-2, September 1991

GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

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Abstract

In the present paper, for special interest, we shall derive generating relations involving multiple hypergeometric functions of four variables introduced by Exton [5, 6, 7].

1. Introduction

Chandel [1] established generating relations for Exton's multiple hypergeometric function ${}^{(k)}_{(1)}E^{(n)}_{D}$ [4] related to Lauricella's $F^{(n)}_{D}$, and for his own multiple hypegeometric function ${}^{(k)}_{(1)}E^{(n)}_{C}$ [1] related to Lauricella's $F^{(n)}_{C}$. Also Chandel and Gupta [2] introduced three intermediate Lauricella function ${}^{(k)}F^{(n)}_{AC}$, ${}^{(k)}F^{(n)}_{AD}$, ${}^{(k)}F^{(n)}_{BD}$ and obtained generating relations involving them. Recently Chandel and Vishwakarma [3] introduced confluent hypergeometric functions of fourth possible intermediate Lauricella's hypergeometric function ${}^{(k)}F^{(n)}_{CD}$ of Karlsson [8] and obtained their generating relations.

In the present paper, for special interest we shall derive generating relations for multiple hypergeometric functions of four variables introduced by Exton [5, 6, 7]. Applying same techniques we can also obtain generating relations for hypergeometric functions of four variables recently introduced by Sharma and Parihar [9].

2. Generating Relations

In this section, we shall derive some interesting generating relations involving multiple hypergeometric functions of four variables $k_1, \ldots, k_{21}$ of Exton [5, 6, 7].

Consider

$$(1-t)^{-a} k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{m,\, n,\, p,\, q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n+p} (c)_q}{(d)_{m+q} (e_1)_n (e_2)_p} \frac{x^m}{m\,!} \frac{y^n}{n\,!} \frac{z^p}{p\,!} \frac{u^q}{q\,!} (1-t)^{-(a+m+n+p+q)}$$

$$= \sum_{m,\, n,\, p,\, q,\, r=0}^{\infty} \frac{(a)_{m+n+p+q+r} (b)_{m+n+p} (c)_q}{(d)_{m+q} (e_1)_n (e_2)_p} \frac{x^m}{m\,!} \frac{y^n}{n\,!} \frac{z^p}{q\,!} \frac{u^q}{p\,!} \frac{t^r}{r\,!}$$

$$= \sum_{r,\, m,\, n,\, p,\, q=0}^{\infty} \frac{(a)_r \, t^r}{r\,!} \frac{(a+r)_{m+n+p+q} (b)_{m+n+p} (c)_q}{(d)_{m+q} (e_1)_n (e_2)_p} \frac{x^m}{m\,!} \frac{y^n}{n\,!} \frac{z^p}{p\,!} \frac{u^q}{q\,!}.$$

Therefore, we establish

$$(1-t)^{-a} k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r\,!} k_1 (a+r, a+r, a+r, a+r; b, b, b, c; d, e_1, e_2, d; x, y, z, u). \qquad \ldots(2.1)$$

Similarly, applying the same techniques, we also obtain the following generating relations :

$$(1-t)^{-b} k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} (b)_r \frac{t^r}{r\,!} k_1 (a, a, a, a; b+r, b+r, b+r, c; d, e_1, e_2, d; x, y, z, u), \qquad \ldots(2.2)$$

$$(1-t)^{-c} k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (c)_r \frac{t^r}{r\,!} k_1 (a, a, a, a; b, b, b, c+r; d, e_1, e_2, d; x, y, z, u), \qquad \ldots(2.3)$$

$$(1-t)^{-a} k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r\,!} k_2 (a+r, a+r, a+r, a+r; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, u), \qquad \ldots(2.4)$$

Consider

$$(1-t)^{-a}\, k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{m,\,n,\,p,\,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b)_{m+n+p}\,(c)_q}{(d)_{m+q}\,(e_1)_n\,(e_2)_p} \frac{x^m}{m\,!}\frac{y^n}{n\,!}\frac{z^p}{p\,!}\frac{u^q}{q\,!}(1-t)^{-(a+m+n+p+q)}$$

$$= \sum_{m,\,n,\,p,\,q,\,r=0}^{\infty} \frac{(a)_{m+n+p+q+r}\,(b)_{m+n+p}\,(c)_q}{(d)_{m+q}\,(e_1)_n\,(e_2)_p} \frac{x^m}{m\,!}\frac{y^n}{n\,!}\frac{z^p}{q\,!}\frac{u^q}{p\,!}\frac{t^r}{r\,!}$$

$$= \sum_{r,\,m,\,n,\,p,\,q=0}^{\infty} \frac{(a)_r\, t^r}{r\,!} \frac{(a+r)_{m+n+p+q}\,(b)_{m+n+p}\,(c)_q}{(d)_{m+q}\,(e_1)_n\,(e_2)_p} \frac{x^m}{m\,!}\frac{y^n}{n\,!}\frac{z^p}{p\,!}\frac{u^q}{q\,!}.$$

Therefore, we establish

$$(1-t)^{-a}\, k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r\,!}\, k_1\,(a+r, a+r, a+r, a+r; b, b, b, c; d, e_1, e_2, d; x, y, z, u). \qquad \text{...(2.1)}$$

Similarly, applying the same techniques, we also obtain the following generating relations :

$$(1-t)^{-b}\, k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} (b)_r \frac{t^r}{r\,!}\, k_1\,(a, a, a, a; b+r, b+r, b+r, c; d, e_1, e_2, d; x, y, z, u), \qquad \text{...(2.2)}$$

$$(1-t)^{-c}\, k_1\left(a, a, a, a; b, b, b, c; d, e_1, e_2, d; x, y, z, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (c)_r \frac{t^r}{r\,!}\, k_1\,(a, a, a, a; b, b, b, c+r; d, e_1, e_2, d; x, y, z, u), \qquad \text{...(2.3)}$$

$$(1-t)^{-a}\, k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{t^r}{r\,!}\, k_2\,(a+r, a+r, a+r, a+r; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, u), \qquad \text{...(2.4)}$$

$$(1-t)^{-b}\, k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$\frac{(b)_r}{r\,!}\; t^r\, k_2\,(a, a, a, a; b+r, b+r, b+r, c; d_1, d_2, d_3, d_4, x, y, z, u). \qquad \ldots(2.5)$$

$$(1-t)^{-c}\, k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, \frac{u}{1-t}\right)$$

$$\frac{(c)_r}{r\,!}\; t^r\, k_2\,(a, a, a, a; b, b, b, c+r; d_1, d_2, d_3, d_4; x, y, z, u), \qquad \ldots(2{\cdot}6)$$

$$(1-t)^{-a}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$\sum_{r=0} (a)_r\, \frac{t^r}{r\,!}\, k_3\,(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.7)$$

$$(1-t)^{-b_1}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$\sum_{r=0} \frac{(b_1)_r}{r\,!}\; t^r\, k_3\,(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.8)$$

$$(1-t)^{-b_2}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$\sum_{r=0}^{\infty} \frac{(b_2)_r}{r\,!}\; t^r\, k_3\,(a, a, a, a; b_1, b_1, b_2+r, b_2+r; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.9)$$

$$(1-t)^{-a}\, k_4\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\; k_4\,(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c, d_1, d_2, c; x, y, z, u), \qquad \ldots(2.10)$$

$$(1-t)^{-b_1}\, k_4\left(a, a, a, a; b_1, b_1, b_2, b_2; c, d_1, d_2, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\; t^r\, k_4\,(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c, d_1, d_2, c; x, y, z, u), \qquad \ldots(2.11)$$

$$(1-t)^{-b}\, k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b)_r}{r\,!}\, t^r\, k_2\,(a, a, a, a; b+r, b+r, b+r, c; d_1, d_2, d_3, d_4, x, y, z, u). \qquad \ldots(2.5)$$

$$(1-t)^{-c}\, k_2\left(a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x, y, z, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(c)_r}{r\,!}\, t^r\, k_2\,(a, a, a, a; b, b, b, c+r; d_1, d_2, d_3, d_4; x, y, z, u), \qquad \ldots(2{\cdot}6)$$

$$(1-t)^{-a}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} (a)_r\, \frac{t^r}{r\,!}\, k_3\,(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.7)$$

$$(1-t)^{-b_1}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_3\,(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.8)$$

$$(1-t)^{-b_2}\, k_3\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_2)_r}{r\,!}\, t^r\, k_3\,(a, a, a, a; b_1, b_1, b_2+r, b_2+r; c_1, c_2, c_2, c_1; x, y, z, u), \qquad \ldots(2.9)$$

$$(1-t)^{-a}\, k_4\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_2, c_1; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, k_4\,(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c, d_1, d_2, c; x, y, z, u), \qquad \ldots(2.10)$$

$$(1-t)^{-b_1}\, k_4\left(a, a, a, a; b_1, b_1, b_2, b_2; c, d_1, d_2, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_4\,(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c, d_1, d_2, c; x, y, z, u), \qquad \ldots(2.11)$$

$$(1-t)^{-b_2} k_4\left(a, a, a, a; b_1, b_1, b_2, b_2; c, d_1, d_2, c; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_2)_r}{r\,!}\, t^r\, k_4\,(a, a, a, a; b_1, b_1, b_2+r, b_2+r; c, d_1, d_2, c; x, y, z, u), \qquad \ldots(2.12)$$

$$(1-t)^{-a}\, k_5\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_3, c_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_5\,(a+r, a+r, a+r, a+r; b_1, b_1, b_2, b_2; c_1, c_2, c_3, c_4; x, y, z, u), \qquad \ldots(2.13)$$

$$(1-t)^{-b_1} k_5\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_3, c_4; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_5\,(a, a, a, a; b_1+r, b_1+r, b_2, b_2; c_1, c_1, c_3, c_4; x, y, z, u), \qquad \ldots(2.14)$$

$$(1-t)^{-b_2}\, k_5\left(a, a, a, a; b_1, b_1, b_2, b_2; c_1, c_2, c_3, c_4; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_2)_r}{r\,!}\, t^r\, k_5\,(a, a, a, a; b_1, b_1, b_2+r, b_2+r; c_1, c_2, c_3, c_4, x, y, z, u), \qquad \ldots(2.15)$$

$$(1-t)^{-a}\, k_6\left(a, a, a, a; b, b, c_1, c_2; e, d, d, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_6\,(a+r, a+r, a+r, a+r; b, b, c_1, c_2; e, d, d, d; x, y, z, u), \qquad \ldots(2.16)$$

$$(1-t)^{-b}\, k_6\left(a, a, a, a; b, b, c_1, c_2; e, d, d, d; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_r)}{r\,!}\, t^r\, k_6\,(a, a, a, a; b+r, b+r, c_1, c_2; e, d, d, d: x, y, z, u), \qquad \ldots(2.17)$$

$$(1-t)^{-c_1}\, k_6\left(a, a, a, a; b, b, c_1, c_2; e, d, d, d; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!}\, t^r\, k_6\,(a, a, a, a; b, b, c_1+r, c_2; e, d, d, d: x, y, z, u), \qquad \ldots(2.18)$$

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$$(1-t)^{-a}\, k_7 \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_7\, (a+r, a+r, a+r, a+r; b, b, c_1, c_2; d_1, d_2, d_1, d_2; x, y, z, u), \quad \ldots(2.19)$$

$$(1-t)^{-b}\, k_7 \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r\,!}\, t^r\, k_7\, (a, a, a, a; b+r, b+r, c_1, c_2; d_1, d_2, d_1, d_2; x, y, z, u), \quad \ldots(2.20)$$

$$(1-t)^{-c_1}\, k_7 \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_1, d_2; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!}\, t^r\, k_7\, (a, a, a, a; b, b, c_1+r\; c_2; d_1, d_2, d_1, d_2; x, y, z, u), \quad \ldots(2.21)$$

$$(1-t)^{-a}\, k_8 \left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_8\, (a+r, a+r, a+r, a+r, b, b, c_1, c_2; d, e_1, d, e_2; x, y, z, u), \quad \ldots(2.22)$$

$$(1-t)^{-b}\, k_8 \left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r\,!}\, t^r\, k_8\, (a, a, a, a; b+r, b+r, c_1, c_2; d, e_1, d, e_2; x, y, z, u), \quad \ldots(2.23)$$

$$(1-t)^{-c_1}\, k_8 \left(a, a, a, a; b, b, c_1, c_2; d, e_1, d, e_2; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!}\, t^r\, k_8\, (a, a, a, a; b, b, c_1+r, c_2; d, e_1, d, e_2; x, y, z, u), \quad \ldots(2.24)$$

$$(1-t)^{-a}\, k_9 \left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_9\, (a+r, a+r, a+r, a+r; b, b, c_1, c_2; e_1, e_2, d, d, x, y, z, u), \quad (2.25)$$

$$(1-t)^{-b} k_9 \left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r\,!} t^r k_9 (a, a, a, a; b+r, b+r, c_1, c_2; e_1, e_2, d, d; x, y, z, u), \quad \ldots(2.26)$$

$$(1-t)^{-c_1} k_9 \left(a, a, a, a; b, b, c_1, c_2; e_1, e_2, d, d; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!} t^r k_9 (a, a, a, a; b, b, c_1+r, c_2; e_1, e_2, d, d; x, y, z, u), \quad \ldots(2.27)$$

$$(1-t)^{-a} k_{10} \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!} t^r k_{10} (a+r, a+r, a+r, a+r; b, b, c_1, c_2; d_1, d_2, d_3, d_4; x, y, z, u),$$

$$\ldots(2.28)$$

$$(1-t)^{-b} k_{10} \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_3, d_4; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r\,!} t^r k_{10} (a, a, a, a; b+r, b+r, c_1, c_2; d_1, d_2, d_3, d_4; x, y, z, u),$$

$$\ldots(2.29)$$

$$(1-t)^{-c_1} k_{10} \left(a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_3, d_4; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!} t^r k_{10} (a, a, a, a; b, b, c_1+r, c_2; d_1, d_2, d_3, d_4; x, y, z, u), \quad \ldots(2.30)$$

$$(1-t)^{-a} k_{11} \left(a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!} t^r k_{11} (a+r, a+r, a+r, a+r; b_1, b_2, b_3, b_4; c, c, c, d; x, y, z, u),$$

$$\ldots(2.31)$$

$$(1-t)^{-b_1} k_{11} \left(a, a, a, a; b_1, b_2, b_3, b_4; c, c, c, d; \frac{x}{1-t}, y, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!} t^r k_{11} (a, a, a, a; b_1+r, b_2, b_3, b_4; c, c, c, d; x, y, z, u), \quad \ldots(2.32)$$

$$(1-t)^{-a} k_{12} (a, a, a, a; b_1, b_2, b_3, b$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!} t^r k_{12} (a+r, a+r, a+r, a+r; b$$

$$(1-t)^{-b_1} k_{12} (a, a, a, a; b_1, b_2, b_3, b$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!} t^r k_{12} (a, a, a, a; b_1+r, b_2, b_3, b$$

$$(1-t)^{-a} k_{13} (a, a, a, a; b_1, b_2, b_3, b_4; c$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!} t^r k_{13} (a+r, a+r, a+r, a+r; b_1,$$

$$(1-t)^{-b_1} k_{13} (a, a, a, a; b_1, b_2, b_3, b_4;$$

$$= \sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!} t^r k_{13} (a, a, a, a; b_1+r, b_2, b_3, b_4;$$

$$(1-t)^{-a} k_{14} (a, a, a, c_3; b, c_1, c_2\, b; d, d,$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!} t^r k_{14} (a+r, a+r, a+r; c_3; b, c_1, c_2$$

$$(1-t)^{-c_3} k_{14} (a, a, a, c_3; b, c_1, c_2, b; d,$$

$$= \sum_{r=0}^{\infty} \frac{(c_3)_r}{r\,!} t^r k_{14} (a, a, a, c_3+r; b, c_1, c_2, b; d,$$

$$(1-t)^{-b} k_{14} (a, a, a, c_3; b, c_1, c_2, b; d, d,$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r\,!} t^r k_{14} (a, a, a, c_3; b+r, c_1, c_2, b+r; d,$$

$$(1-t)^{-a}\, k_{12}\left(a, a, a, a; b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_{12}\,(a+r, a+r, a+r, a+r; b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2; x, y, z, u), \qquad \ldots(2.33)$$

$$(1-t)^{-b_1}\, k_{12}\left(a, a, a, a; b_1. b_2, b_3, b_4; c_1, c_1, c_2, c_2; \frac{x}{1-t}, y, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{12}\,(a, a, a, a; b_1+r, b_2, b_3, b_4; c_1, c_1, c_2, c_2; x, y, z, u), \qquad \ldots(2.34)$$

$$(1-t)^{-a}\, k_{13}\left(a, a, a, a; b_1, b_2, b_3, b_4; c, c, d_1, d_2; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_{13}\,(a+r, a+r, a+r, a+r; b_1, b_2, b_3, b_4; c, c, d_1, d_2; x, y, z, u), \qquad \ldots(2.35)$$

$$(1-t)^{-b_1}\, k_{13}\left(a, a, a, a; b_1, b_2, b_3, b_4; c, c, d_1, d_2; \frac{x}{1-t}, y, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{13}\,(a, a, a, a; b_1+r, b_2, b_3, b_4; c, c, d_1, d_2; x, y, z, u), \qquad \ldots(2.36)$$

$$(1-t)^{-a}\, k_{14}\left(a, a, a, c_3; b, c_1, c_2\, b; d, d, d, d; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_{14}\,(a+r, a+r, a+r; c_3; b, c_1, c_2, b; d, d, d, d; x, y, z, u), \qquad \ldots(2.37)$$

$$(1-t)^{-c_3}\, k_{14}\left(a, a, a, c_3; b, c_1, c_2, b; d, d, d, d; x, y, z, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(c_3)_r}{r\,!}\, t^r\, k_{14}\,(a, a, a, c_3+r; b, c_1, c_2, b; d, d, d, d; x, y, z, u), \qquad \ldots(2.38)$$

$$(1-t)^{-b}\, k_{14}\left(a, a, a, c_3; b, c_1, c_2, b; d, d, d, d; \frac{x}{1-t}, y, z, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b)_r}{r\,!}\, t^r\, k_{14}\,(a, a, a, c_3; b+r, c_1, c_2, b+r; d, d, d, d; x, y, z, u), \qquad \ldots(2.39)$$

$$(1-t)^{-c_1} k_{14}\left(a, a, a, c_3, b, c_1, c_2, b; d, d, d, d; x, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(c_1)_r}{r\,!}\, t^r\, k_{14}\,(a, a, a, c_3, b, c_1+r, b; d, d, d, d; x, y, z, u), \qquad \ldots(2.40)$$

$$(1-t)^{-a} k_{15}\left(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_{15}\,(a+r, a+r, a+r, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \qquad \ldots(2.41)$$

$$(1-t)^{-b_5} k_{15}\left(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_5)_r}{r\,!}\, t^r\, k_{15}\,(a, a, a, b_5+r; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \qquad \ldots(2.42)$$

$$(1-t)^{-b_1} k_{15}\left(a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; \frac{x}{1-t}, y, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{15}\,(a, a, a, b_5; b_1+r, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \qquad \ldots(2.43)$$

$$(1-t)^{-a_1} k_{16}\left(a_1, a_2, a_3, a_4; b; \frac{x}{1-t}, \frac{y}{1-t}, z. u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_1)_r}{r\,!}\, t^r\, k_{16}\,(a_1+r, a_2, a_3, a_4; x, y, z, u), \qquad \ldots(2.44)$$

$$(1-t)^{-a_2} k_{16}\left(a_1, a_2, a_3, a_4; b; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_2)_r}{r\,!}\, t^r\, k_{16}\,(a_1, a_2+r, a_3, a_4; b; x, y, z, u), \qquad \ldots(2.45)$$

$$(1-t)^{-a_3} k_{16}\left(a_1, a_2, a_3, a_4; b; x, \frac{y}{1-t}, y, \frac{z}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_3)_r}{r\,!}\, t^r\, k_{16}\,(a_1, a_2, a_3+r, a_4; b; x, y, z, u), \qquad \ldots(2.46)$$

$$(1-t)^{-a_4}\, k_{16}\left(a_1, a_2, a_3, a_4; b; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_4)_r}{r\,!}\, t^r\, k_{16}\,(a_1, a_2, a_3, a_4+r; b; x, y, z, u), \qquad \ldots(2.47)$$

$$(1-t)^{-a_1}\, k_{17}\left(a_1, a_2, a_3, b_1, b_2; c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_1)_r}{r\,!}\, t^r\, k_{17}\,(a_1+r, a_2, a_3, b_1, b_2; c; x, y, z, u), \qquad \ldots(2.48)$$

$$(1-t)^{-a_2}\, k_{17}\left(a_1, a_2, a_3, b_1, b_2; c; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_2)_r}{r\,!}\, t^r\, k_{17}\,(a_1, a_2+r, a_3, b_1, b_2; c; x, y, z, u), \qquad \ldots(2.49)$$

$$(1-t)^{-a_3}\, k_{17}\left(a_1, a_2, a_3, b_1, b_2; c; x, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_3)_r}{r\,!}\, t^r\, k_{17}\,(a_1, a_2, a_3+r, b_1, b_2; x, y, z, u), \qquad \ldots(2.50)$$

$$(1-t)^{-b_1}\, k_{17}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{17}\,(a_1, a_2, a_3, b_1+r, b_2; c; x, y, z, u), \qquad \ldots(2.51)$$

$$(1-t)^{-a_1}\, k_{18}\left(a_1, a_2, a_3, b_1, b_2; c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_1)_r}{r\,!}\, t^r\, k_{18}\,(a_1+r, a_2, a_3; b_1, b_2; c; x, y, z, u), \qquad \ldots(2.52)$$

$$(1-t)^{-a_2}\, k_{18}\left(a_1, a_2, a_3, b_1, b_2; c; \frac{x}{x-1}, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_2)_r}{r\,!}\, t^r\, k_{18}\,(a_1, a_2+r, a_3, b_1, b_2; c; x, y, z, u), \qquad \ldots(2.53)$$

$$(1-t)^{-a_3}\, k_{18}\left(a_1, a_2, a_3, b_1, b_2; c; x, \frac{y}{1-t}, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_3)_r}{r\,!}\, t^r\, k_{18}\,(a_1, a_2, a_3+r, b_1, b_2; c; x, y, z, u), \qquad \ldots(2.54)$$

$$(1-t)^{-b_1}\, k_{18}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{18}\,(a_1, a_2, a_3, b_1+r, b_2; c; x, y, z, u), \qquad \ldots(2.55)$$

$$(1-t)^{-a_1}\, k_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_1)_r}{r\,!}\, t^r\, k_{19}\,(a_1+r, a_2, b_1, b_2, b_3, b_4; x, y, z, u), \qquad \ldots(2.56)$$

$$(1-t)^{-a_2}\, k_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; \frac{x}{1-t}, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_2)_r}{r\,!}\, t^r\, k_{19}\,(a_1, a_2+r, b_1, b_2, b_3, b_4; c; x, y, z, u). \qquad \ldots(2.57)$$

$$(1-t)^{-b_1}\, k_{19}\left(a_1, a_2, b_1, b_2, b_3, b_4; c; x, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_1)_r}{r\,!}\, t^r\, k_{19}\,(a_1, a_2, b_1+r, b_2, b_3, b_4; c; x, y, z, u), \qquad \ldots(2.58)$$

$$(1-t)^{-a_1}\, k_{20}\left(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_1)_r}{r\,!}\, t^r\, k_{20}\,(a_1+r, a_1+r, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; x, y, z, u), \qquad \ldots(2.59)$$

$$(1-t)^{-a_2}\, k_{20}\left(a_1, a_1, b_3, b_4; b_1, b_2, a_2, a_2; c, c, c, c; x, y, \frac{z}{1-t}, \frac{u}{1-t}\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a_2)_r}{r\,!}\, t^r\, k_{20}\,(a_1, a_1, b_3, b_4; b_1, b_2, a_2+r, a_2+r; c, c, c, c; x, y, z, u), \qquad \ldots(2.60)$$

$$(1-t)^{-a}\, k_{21}\left(a, a, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r\,!}\, t^r\, k_{21}\,(a+r, a+r, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \quad \ldots(2.61)$$

$$(1-t)^{-b_5}\, k_{21}\left(a, a, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; x, y, \frac{z}{1-t}, u\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b_5)_r}{r\,!}\, t^r\, k_{21}\,(a, a, b_5+r, b_6, b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \quad \ldots(2.62)$$

Applying the same techniques, we can also obtain generating relation for the hypergeometric functions of four variables recently introduced by Sharma and Parihar [9].

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Received : October 26, 1990
Revised : January 7, 1991
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$$(1-t)^{-a}\, k_{21}\left(a, a, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; \frac{x}{1-t}, \frac{y}{1-t}, z, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\, k_{21}\,(a+r, a+r, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \qquad (2.61)$$

$$(1-t)^{-b_5}\, k_{21}\left(a, a, b_5, b_6; b_1, b_2, b_3, b_4; c, c, c, c; x, y, \frac{z}{1-t}, u\right)$$

$$=\sum_{r=0}^{\infty} \frac{(b_5)_r}{r!}\, t^r\, k_{21}\,(a, a, b_5+r, b_6, b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, u), \quad \ldots(2.62)$$

Applying the same techniques, we can also obtain generating relation for the hypergeometric functions of four variables recently introduced by Sharma and Parihar [9].

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Proc. VPI. Vol 2, 1990 112

MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST AND ITS APPLICATIONS IN TWO BUUNDARY VALUE PROBLEMS

by

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In the present paper first we evaluate an interesting integral involving multiple hypergeometric function of Srivastava and Daoust [*Publ Inst. Math.* (Beograd) (N. S.). 9 (23) (1969), 199-202, *Nederl. Acad. Wetensch. Proc. Ser. A* 72 = Indag Math. 31 (1969), 449-457; *Math. Nachr.* 53 (1972), 151-159] and then we make its applications to solve two boundary value problems on (i) heat conduction in a rod (ii) deflection of a vibrating string, under certain conditions and to establish an expansion formula involving above multiple hypergeometric function.